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THE CONVERSE OF FERMAT'S THEOREM

RAPHAEL M. ROBINSON, University of California, Berkeley

1. **Basic theorems.** In this paper, we discuss some of the known results concerning the use of the converse of Fermat's theorem as a test for primeness, and prove several results, believed to be new, which clarify certain points and also delimit to some extent the nature of possible improvements. The proofs of the known results are included, so that only the standard material of elementary number theory is presupposed.

Fermat's theorem states that if N is prime and $(a, N)=1$, then

$$a^{N-1} \equiv 1 \pmod{N}.$$

Alternatively, we may say that $a^N \equiv a \pmod{N}$ for any value of a, but we prefer the first form. It is well known that this congruence is not a sufficient condition for N to be prime, even if we exclude trivial cases such as $a=\pm 1$. For example, we have $2^{10}-1=3\cdot 341$, hence $2^{340}\equiv 1 \pmod{341}$, although $341=11\cdot 31$ is not prime.

An interesting class of numbers, not all prime, satisfying Fermat's congruence with $a=2$, was given by Malo [9], namely the Mersenne numbers. If n is prime, then $2^n \equiv 2 \pmod{n}$. Putting $N=2^n-1$, we have $2^n \equiv 1 \pmod{N}$ and $N \equiv 1 \pmod{n}$, hence $2^{N-1}\equiv 1 \pmod{N}$. As pointed out by Sierpiński [16], we do not use the fact that n is prime, but only that $2^n \equiv 2 \pmod{n}$. Thus if n satisfies the congruence $2^{x-1}\equiv 1 \pmod{x}$, so also does $N=2^n-1$. Furthermore, if n is composite, so also is N. This furnishes a simple proof that there are infinitely many composite solutions.

Despite this, Fermat's congruence is important for testing numbers for primeness. It is of course a necessary condition, and it serves to eliminate most composite numbers. One possibility, used, for example, by Lehmer [5], [6], is to make a list of the exceptional cases in a certain range. But, so far as possible, it is desirable to give supplementary conditions which insure that N is prime, or, failing that, at least to give restrictions on the form of the factors.

The following three theorems are of that type. Theorem 1 is essentially contained in Lucas [7], p. 66, and [8], pp. 161–162. (Other references to the early literature may be found in Dickson [1], both in the section on the converse of Fermat's theorem, pp. 91–95, and in the chapter on Fermat numbers, pp. 375–380.) Theorem 2 is due to Lehmer [2] and [3]. I have not seen Theorem 3 in the form given, though of course it is closely related to the known results.

It should perhaps be emphasized that such direct tests for primeness are applicable to much larger numbers than one can be sure of factoring by trial. Numbers as large as $2^{2281}-1$ have been identified as prime by the author [14], whereas much smaller numbers, such as $2^{101}-1$ (which is known to be composite), have resisted factorization. In both cases, use was made of high-speed computers.

THEOREM 1 (Lucas). *If* $N>1$ *and*

$$a^{N-1} \equiv 1 \pmod{N},$$

but

$$a^{(N-1)/q} \not\equiv 1 \pmod{N}$$

for each prime factor q *of* $N-1$, *then* N *is prime.*

Proof. It follows from the hypotheses that a belongs to the exponent $N-1$ mod N, and hence that $N-1 \mid \phi(N)$. But if N were composite, we would have $\phi(N) < N-1$.

THEOREM 2 (Lehmer). *Let* $N = kq^n+1$, *where* $k>0$, $n>0$, *and* q *is prime. Suppose that*

$$a^{N-1} \equiv 1 \pmod{N}.$$

Then every prime factor p *of* N *which does not divide* $a^{(N-1)/q}-1$ *satisfies the congruence*

$$p \equiv 1 \pmod{q^n}.$$

In particular, if $(a^{(N-1)/q}-1, N) = 1$, *then every prime factor* p *of* N *satisfies this congruence.*

Proof. Suppose that a belongs to the exponent d mod p. Then $d \mid N-1$ but $d \nmid (N-1)/q$, that is, $d \mid kq^n$ but $d \nmid kq^{n-1}$. It follows that $q^n \mid d$. Since $d \mid p-1$, we see that $q^n \mid p-1$.

THEOREM 3. *Let* $N = kQ+1$, *where* $0<k<Q$. *Suppose that*

$$a^{N-1} \equiv 1 \pmod{N},$$

but

$$(a^{(N-1)/q} - 1, N) = 1$$

for every prime factor q *of* Q. *Then* N *is prime.*

Proof. If p and q are primes such that $p \mid N$ and $q^n \mid Q$, then by Theorem 2 we have $p \equiv 1 \pmod{q^n}$. It follows that $p \equiv 1 \pmod{Q}$. Thus

$$p^2 > Q^2 \geqq (k+1)Q > N.$$

That is, for every prime $p \mid N$ we have $p^2 > N$. It follows that N is prime.

Remark. Comparing Theorems 1 and 3, we see that the former theorem requires, for each prime divisor q of $N-1$, that $a^{(N-1)/q}-1$ should not be a multiple of N, whereas the latter imposes the more stringent condition that this quantity should be prime to N, but for fewer values of q. If we can write N in the form $N = kq^n+1$ with $0<k<q^n$, then only this single value of q need be considered.

The most striking advantage of Theorem 3 over Theorem 1 is that it does not require the complete factorization of $N-1$ to be known. However, we still require a partial factorization in which the factored portion of $N-1$ exceeds the unfactored portion.

It may be remarked that there is no known practical method of testing a number N for primeness, if N is too large to be factored by trial which is free from such a restriction. It is possible, however, by using the Lucas functions, to make use of a factorization of $N+1$ instead of $N-1$; see, for example, Lehmer [4] and Robinson [14].

2. A more precise result. In this section, we prove a theorem which on the one hand extends Theorem 2, and on the other shows that no improvement of a certain type is possible.

LEMMA. *Suppose that a belongs to the exponent d mod p^l, where p is prime and $l>0$. If $p\nmid d$, then a also belongs to the exponent d mod p.*

Proof. Suppose that a belongs to the exponent e mod p. Then clearly $e \mid d$. On the other hand, by raising the congruence $a^e \equiv 1 \pmod{p}$ to the pth power $l-1$ times, we find that

$$a^{ep^{l-1}} \equiv 1 \pmod{p^l},$$

since the exponent of p in the modulus can be increased by one unit each time. Hence $d \mid ep^{l-1}$. Since $p \nmid d$, this yields $d \mid e$, and hence $d=e$.

THEOREM 4. *Let $N=kq^n+1$, where $k>0$, $n>0$, q is prime, and $q\nmid k$. Then there exists a number a such that*

$$a^{N-1} \equiv 1 \pmod{N}$$

and

$$(a^{(N-1)/q} - 1, N) = \delta,$$

if and only if $\delta \mid N$, $(\delta, N/\delta)=1$, and every prime factor p of N/δ satisfies the congruence

$$p \equiv 1 \pmod{q^n}.$$

Proof. Suppose first that such an a exists. Then clearly $\delta \mid N$. Now choose any prime p such that $p \mid \delta$, and take the largest l such that $p^l \mid N$. Suppose that a belongs to the exponent d mod p^l. Then $d \mid N-1$, hence $p \nmid d$. Thus, by the lemma, a also belongs to the exponent d mod p. Since $a^{(N-1)/q} \equiv 1 \pmod{p}$, we will also have $a^{(N-1)/q} \equiv 1 \pmod{p^l}$, so that $p^l \mid \delta$. Hence $p \nmid N/\delta$, and therefore $(\delta, N/\delta)=1$.

Next choose any prime p which divides N/δ. Then $p \nmid \delta$, that is, $p \nmid a^{(N-1)/q}-1$, and hence, by Theorem 2, we have $p \equiv 1 \pmod{q^n}$.

Now consider the converse. Supposing that the stated conditions are satisfied, we must show how to choose a. The value of a mod p^l may be assigned

independently for each prime power p^l in the canonical factorization of N. The required conditions are

$$a^{(N-1)/q} \equiv 1 \pmod{p^l} \text{ if } p^l \mid \delta,$$

$$a^{N-1} \equiv 1 \pmod{p^l}, \text{ but } a^{(N-1)/q} \not\equiv 1 \pmod{p}, \text{ if } p^l \mid N/\delta.$$

The first condition may be satisfied by taking $a \equiv 1 \pmod{p^l}$ whenever $p^l \mid \delta$, which amounts to supposing that $a \equiv 1 \pmod{\delta}$. If $p^l \mid N/\delta$, it will be sufficient to choose for the value of $a \bmod p^l$ a number which belongs to the exponent $q^n \bmod p^l$. This is possible since $q^n \mid p-1$, hence $p>2$, so that there is a primitive root mod p^l, and hence a number belonging to any prescribed exponent which divides $\phi(p^l)$, for example q^n. Now since $p \nmid q^n$, we see that p also belongs to the exponent $q^n \bmod p$. Since $q^n \mid N-1$, but $q^n \nmid (N-1)/q$, we see that the required congruence and incongruence are both satisfied.

Remark. Theorem 4 gives an improvement of Theorem 2, in that the conclusion $p \equiv 1 \pmod{q^n}$ is drawn for all prime factors p of N/δ rather than for the apparently narrower class of factors of N not dividing δ. This result was once stated by Lehmer [3], Theorem 4, but was later weakened to that given in Theorem 2 above (see [4], p. 443, footnote), which was all that his proof justified. Of course, the proof shows that the two classes are actually the same.

The most interesting aspect of Theorem 4, however, is the converse. Restricting ourselves to the simplest and most usual case $\delta = 1$, the theorem states that there exists a number a such that $a^{N-1} \equiv 1 \pmod{N}$, but $a^{(N-1)/q} - 1$ is prime to N, if and only if every prime factor p of N satisfies the congruence $p \equiv 1 \pmod{q^n}$. Thus the last sentence of Theorem 2 draws the strongest possible conclusion from the mere existence of such an a. Of course, if the value of a is known, further conclusions might be drawn. One example of such a conclusion is given in Theorem 6 below.

3. Euler's criterion. According to Euler, if N is prime, then

$$a^{(N-1)/2} \equiv (a/N) \pmod{N},$$

where on the right we have a Legendre symbol. This is usually considered as a criterion for a to be a quadratic residue mod N. Interpreting the right side as a Jacobi symbol, we shall see that this congruence is also satisfied for some composite values of N. Notice that, for $(a, N) = 1$, it implies $a^{N-1} \equiv 1 \pmod{N}$, so that it is certainly as strong a condition on N as Fermat's congruence. It is in fact more restrictive. We shall find that it is very useful in testing N for primeness, at least when $(a/N) = -1$. We shall study this case in some detail.

THEOREM 5. *Let $N = k \cdot 2^n + 1$, where $k>0$, $n>0$, and k is odd. Then there exists an a such that*

$$a^{(N-1)/2} \equiv -1 \pmod{N},$$

if and only if every prime factor p of N satisfies the congruence $p \equiv 1 \pmod{2^n}$.

Proof. If a satisfies the condition stated, then $a^{N-1}\equiv 1 \pmod{N}$ and $(a^{(N-1)/2}-1, N)=1$. The conclusion follows from Theorem 2. Conversely, by Theorem 4, if the prime factors of N satisfy the given condition, then some a will satisfy the conditions just mentioned. Thus for each prime power $p^l \mid N$ we will have $a^{N-1}\equiv 1 \pmod{p^l}$ but $a^{(N-1)/2}\not\equiv 1 \pmod{p^l}$, hence (since p is odd) $a^{(N-1)/2}\equiv -1 \pmod{p^l}$, and so $a^{(N-1)/2}\equiv -1 \pmod{N}$.

Remark. Thus from the existence of a such that $a^{(N-1)/2}\equiv -1 \pmod{N}$, we can conclude only that every prime factor of N satisfies the condition $p\equiv 1 \pmod{2^n}$. However if the value of a is known, it may be possible to draw further conclusions. The next theorem furnishes an example of this.

THEOREM 6. *Let $N=k\cdot 2^n+1$, where $k>0$, $n>0$, and k is odd. Suppose that*

$$a^{(N-1)/2} \equiv -1 \pmod{N}.$$

Then for every prime factor p of N we have not only $p\equiv 1 \pmod{2^n}$, but also

$$(a/p) = (-1)^{(p-1)/2^n}.$$

Proof. Suppose that a belongs to the exponent d mod p. Then $d \mid N-1$ but $d \nmid (N-1)/2$, hence d is an odd multiple of 2^n, and thus also $2^n \mid p-1$ (as previously shown). We must have

$$a^{d/2} \equiv -1 \pmod{p},$$

and hence, raising both sides to the power $(p-1)/d$,

$$(a/p) \equiv a^{(p-1)/2} \equiv (-1)^{(p-1)/d} \pmod{p}.$$

Thus

$$(a/p) = (-1)^{(p-1)/d} = (-1)^{(p-1)/2^n}.$$

THEOREM 7. *If $N>1$ is odd, and*

$$a^{(N-1)/2} \equiv -1 \pmod{N},$$

then $(a/N)=-1$.

Proof. We may put $N=k\cdot 2^n+1$, where $k>0$, $n>0$, and k is odd. Let the canonical factorization of N be

$$N = p_1^{l_1} p_2^{l_2} \cdots p_r^{l_r}.$$

Put $p_i-1=2^n s_i$. Then

$$N = \prod_{i=1}^{r} p_i^{l_i} = \prod_{i=1}^{r} (1+2^n s_i)^{l_i},$$

hence

$$N \equiv 1 + 2^n \sum_{i=1}^{r} l_i s_i \pmod{2^{2n}}.$$

It follows that $\sum l_i s_i$ is odd. Now by Theorem 6, we have $(a/p_i) = (-1)^{s_i}$. Hence

$$(a/N) = \prod_{i=1}^{r} (a/p_i)^{l_i} = \prod_{i=1}^{r} (-1)^{l_i s_i} = -1.$$

Remark. On the other hand, from $a^{(N-1)/2} \equiv 1 \pmod{N}$, it does not follow that $(a/N) = 1$, as the example $a = 2$, $N = 341$ shows.

THEOREM 8. *If* $N = (a^{2^m}+1)/2$, *where* $a \equiv \pm 3 \pmod{8}$ *and* $m > 0$, *then* $a^{(N-1)/2} \equiv -1 \pmod{N}$.

Proof. Since $a^2 \equiv 1 \pmod{8}$, but $a^2 \not\equiv 1 \pmod{16}$, we find by squaring $m-1$ times that

$$a^{2^{m-1}} \equiv 1 \pmod{2^{m+2}}, \qquad a^{2^{m-1}} \not\equiv 1 \pmod{2^{m+3}},$$

since the permissible power of 2 in the modulus increases by just one unit each time. Thus $a^{2^{m-1}} = k \cdot 2^{m+2} + 1$, where k is odd, and hence $N = k \cdot 2^{m+1} + 1$. Now

$$a^{2^{m-1}} = 2N - 1 \equiv -1 \pmod{N},$$

hence

$$a^{(N-1)/2} = a^{k \cdot 2^{m-1}} \equiv (-1)^k \equiv -1 \pmod{N}.$$

Remark. This theorem furnishes an interesting example of a class of numbers N which are not all prime, but all of which nevertheless satisfy Euler's criterion. In particular, for $a = 3$ and $m = 3$, we find that the composite number

$$N = 3281 = 205 \cdot 2^4 + 1 = (2^4 + 1)(12 \cdot 2^4 + 1) = 17 \cdot 193$$

satisfies the congruence $3^{(N-1)/2} \equiv -1 \pmod{N}$.

4. Tests for primeness using Euler's criterion. The following theorem was stated by Proth [13], but without any indication of a proof. The special case $k = 1$ had been proved earlier by Pepin [12].

THEOREM 9 (Proth). *Let* $N = k \cdot 2^n + 1$, *where* $0 < k < 2^n$. *Suppose that* $(a/N) = -1$. *Then* N *is prime, if and only if*

$$a^{(N-1)/2} \equiv -1 \pmod{N}.$$

Proof. If N is prime, then the congruence is satisfied, according to Euler. Now consider the converse. We may suppose that k is odd. By Theorem 5, every prime factor p of N satisfies the congruence $p \equiv 1 \pmod{2^n}$. Hence $p^2 > 2^{2n} \geqq (k+1) \cdot 2^n > N$. It follows that N is prime. This conclusion can also be drawn by specializing Theorem 3.

COROLLARY. *Let* $N=k\cdot 2^n+1$, *where* $n>1$, $0<k<2^n$, *and* $3\nmid k$. *Then* N *is prime, if and only if*

$$3^{(N-1)/2} \equiv -1 \pmod{N}.$$

Proof. This is trivial if $3 \mid N$, since $N>3$ is not prime and the congruence is false. Otherwise, we must have $N\equiv 2 \pmod 3$ as well as $N\equiv 1 \pmod 4$, hence $(3/N)=(N/3)=(2/3)=-1$.

Remark. Thus, in many cases, we can take $a=3$ in Theorem 9. But in any case, a suitable a is easily found. This criterion is a very practical test for primeness, and when applicable appears to be the easiest known test. It was used (in the Pepin special case) by Morehead and Western [10], [11] to show that the Fermat numbers $F_7=2^{128}+1$ and $F_8=2^{256}+1$ are composite, and by the author [14] to show that $F_{10}=2^{1024}+1$ is composite. More recently, the theorem in its general form has been used by the author [15] to identify hundreds of large primes.

We may raise the question, to what extent the restriction $k<2^n$ in Theorem 9 is necessary. The following theorem provides the most complete answer which I have been able to find to this question.

THEOREM 10. *Let* $N=k\cdot 2^n+1$, *where* $n>2$ *and* $0<k<6\cdot 2^n+7$. *Suppose that* $(a/N)=-1$. *Then* N *is prime, if and only if*

$$a^{N-1/2} \equiv 1 \pmod{N}.$$

The conclusion is no longer valid, if we extend the range of k *to* $0<k\leqq 6\cdot 2^n+7$.

Proof. Assuming that the congruence holds, we have to show that N is prime. We may suppose that k is odd. By Theorem 5, every prime factor p of N must satisfy the condition $p\equiv 1 \pmod{2^n}$. Since $N<(2^n+1)^2$, we see that if N is composite, it must be the product of two primes, $N=(x\cdot 2^n+1)(y\cdot 2^n+1)$. It follows that $k=xy\cdot 2^n+x+y$. Hence $x+y$ is odd, and $xy\cdot 2^n+x+y<6\cdot 2^n+7$. We may suppose that x is odd and y is even. Then the only values of x and y satisfying the inequality are

$$x = 1,\ y = 2; \qquad x = 1,\ y = 4; \qquad x = 3,\ y = 2.$$

Now 2^n+1 and $2^{n+1}+1$ cannot both be prime unless $n=1$, and 2^n+1 and $2^{n+2}+1$ cannot both be prime unless $n=2$, since the exponents must be powers of 2. Also, $3\cdot 2^n+1\equiv 0 \pmod 5$, whenever $n+1\equiv 0 \pmod 4$, so that $3\cdot 2^n+1$ and $2^{n+1}+1$ cannot both be prime if $n>1$. Thus in none of the possible cases can $x\cdot 2^n+1$ and $y\cdot 2^n+1$ both be prime if $n>2$. It follows that N must be prime.

On the other hand, if we allow $k=6\cdot 2^n+7$, then for $n=4$ we have

$$N = 103\cdot 2^4 + 1 = (2^4 + 1)(6\cdot 2^4 + 1) = 17\cdot 97,$$

where both factors are prime. By Theorem 5, there must be some value of a such that $a^{(N-1)/2}\equiv -1 \pmod N$. (In fact, this congruence is satisfied if $a=12$.)

By Theorem 7, we will also have $(a/N) = -1$.

Remark. We can further relax the bound on k, when n is sufficiently large, if and only if there are only a finite number of values of n for which both 2^n+1 and $6 \cdot 2^n+1$ are prime. This appears likely, but I cannot prove it.

References

1. L. E. Dickson, History of the Theory of Numbers, vol. 1, 1919.
2. D. H. Lehmer, Tests for primality by the converse of Fermat's theorem, Bull. Amer. Math. Soc., vol. 33, 1927, pp. 327–340.
3. ———, A further note on the converse of Fermat's theorem, *ibid.*, vol. 34, 1928, pp. 54–56.
4. ———, An extended theory of Lucas' functions, Ann. of Math., ser. 2, vol. 31, 1930, pp. 419–448.
5. ———, On the converse of Fermat's theorem, this MONTHLY, vol. 43, 1936, pp. 347–354.
6. ———, On the converse of Fermat's theorem II, *ibid.*, vol. 56, 1949, pp. 300–309.
7. E. Lucas, Sur la recherche des grands nombres premiers, Association Française pour l'Avancement des Sciences, Compte-Rendu, vol. 5, 1876, pp. 61–68.
8. ———, Considérations nouvelle sur la théorie des nombres premiers et sur la division géométrique de la circonférence en parties égales, *ibid.*, vol. 6, 1877, pp. 159–167.
9. E. Malo, Nombres qui, sans être premiers, vérifient exceptionellement une congruence de Fermat, L'Intermédiaire des Mathématiciens, vol. 10, 1903, p. 88.
10. J. C. Morehead, Note on Fermat's numbers, Bull. Amer. Math. Soc., vol. 11, 1905, pp. 543–545.
11. J. C. Morehead and A. E. Western, Note on Fermat's numbers, *ibid.*, vol. 16, 1909, pp. 1–6.
12. T. Pepin, Sur la formule $2^{2^n}+1$, C. R. Acad. Sci. Paris, vol. 85, 1877, pp. 329–331.
13. F. Proth, Théorèmes sur les nombres premiers, *ibid.*, vol. 87, 1878, p. 926.
14. R. M. Robinson, Mersenne and Fermat numbers, Proc. Amer. Math. Soc., vol. 5, 1954, pp. 842–846.
15. ———, Factors of Fermat numbers, Math. Tables Aids Comput., vol. 11, 1957, pp. 21–22.
16. W. Sierpiński, Remarque sur une hypothèse des Chinois concernant les nombres $(2^n-2)/n$, Colloq. Math., vol. 1, 1947, p. 9.

A MOVING BOUNDARY FILTRATION PROBLEM OR "THE CIGARETTE PROBLEM"*

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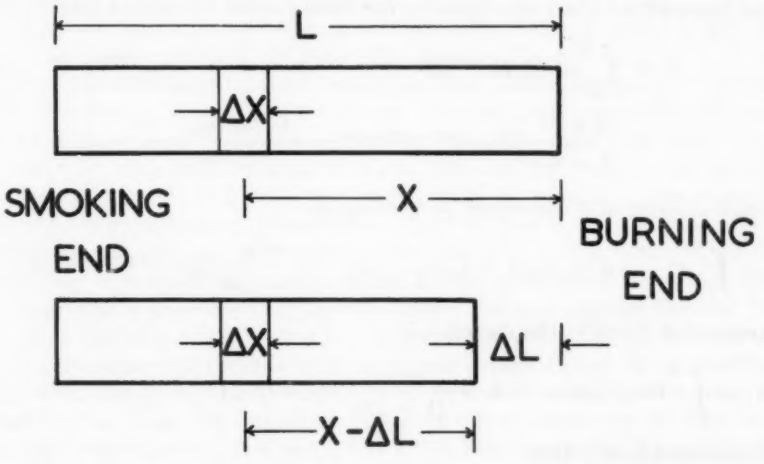
In this paper a study is made of filtration through a burning cigarette. Let us consider a uniform cigarette being smoked under the following conditions:

(1) Steady inhalation.
(2) A constant fraction $1-a$ of any component Z in the tobacco is burnt at the tip (the fraction a being transmitted and filtered down the cigarette).

* Presented at the International Congress of Mathematicians, Cambridge, Mass., Sept., 1950.

(3) The absorption coefficient of the tobacco is constant and equal to b with respect to the filtration of component Z.

Let $C(x, L)$ denote the concentration of component Z in unit weight per unit length for a cigarette of length L at a distance x from the burning end. The



amount of component Z transmitted down the cigarette after burning ΔL of cigarette is, to first order terms, $\Delta A = aC(0, L)\Delta L$. The fraction of ΔA filtered out by element Δx is

$$\Delta(\Delta A) = \Delta C \cdot \Delta x = abe^{-bx}C(0, L)\Delta L \cdot \Delta x. \tag{4}$$

This follows since the filtration law is $dA/dx = -bA$ or $A = A_0e^{-bx}$, and thus, to first order terms, $\Delta A = bA\Delta x$.

Equation (4) can be rewritten as

$$\frac{C(x - \Delta L, L - \Delta L) - C(x, L)}{\Delta L} = abC(0, L)e^{-bx}.$$

On letting $\Delta L \to 0$, we obtain the mixed partial differential equation of first order,

$$\frac{\partial C(x, L)}{\partial x} + \frac{\partial C(x, L)}{\partial L} = -abC(0, L)e^{-bx}, \tag{5}$$

subject to the initial condition $C(x, L_0) = C_0$ (constant), where L_0 is the initial length of the cigarette.

To solve (5) we assume a solution of the form $C(x, L) = C_0 + \phi(L)e^{-bx}$, where $\phi(L_0) = 0$. This transforms (5) into $\phi'(L) - b(1-a)\phi(L) = -abC_0$. Thus,

$$C(x, L) = C_0 + \frac{C_0 a e^{-bx}}{1 - a}\left[1 - e^{-b(1-a)(L_0 - L)}\right]$$

The amount of component Z transmitted to the smoker after burning ΔL of the cigarette is, to first order terms, $\Delta A_t = aC(0, L)e^{-bL}\Delta L$. Thus the total amount transmitted when the cigarette has been burned down to a length L_f is

$$(6) \qquad \begin{aligned} A_t &= \int_{L_f}^{L_0} aC(0, L)e^{-bL}dL \\ &= \frac{aC_0 e^{-bL_f}}{(1 - a)b}\left[1 - e^{-b(1-a)(L_0 - L_f)}\right] = \frac{C(L_f, L_f) - C_0}{b}. \end{aligned}$$

The total amount of Z destroyed by burning is

$$A_b = \int_{L_f}^{L_0} (1 - a)C(0, L)dL = C_0(L_0 - L_f) - \frac{aC_0}{(1 - a)b}\left[1 - e^{-b(1-a)(L_0 - L_f)}\right].$$

The amount of Z left in the cigarette is

$$A_c = \int_0^{L_f} C(x, L_f)dx = C_0 L_f + \frac{aC_0}{(1 - a)b}\left[1 - e^{-bL_f}\right]\left[1 - e^{-b(1-a)(L_0 - L_f)}\right].$$

It follows immediately that

$$(7) \qquad A_t + A_b + A_c = C_0 L_0.$$

Equation (7) expresses the material balance on component Z.

We can now determine the effect of the initial length L_0 on the amount A_t transmitted to the smoker. Let us consider two cigarettes of lengths L_0 and L_0', and identical otherwise. The amounts A_t and A_t' may be compared on one of the following two bases:

(B_1) Both cigarettes burn down the same amount.
(B_2) Both cigarettes burn down to the same final length.

In the latter case, we would consider the amount A_t per unit length of cigarette smoked.

On either basis, the longer cigarette is more effective in reducing the amount A_t transmitted to the smoker. This follows directly from (6) and the fact that $(1-e^{-x})/x$ is a monotonic decreasing function of x for $x>0$.

We shall now consider the effects of filtration alone on the amount A_t. We let $a \to 1$ (zero burning fraction). Then (6) reduces to $A_t = C_0 e^{-bL_f}(L_0 - L_f)$. Using basis ($B_1$) for a comparison we get

$$(8) \qquad \frac{A_t}{A_t'} = \frac{e^{-bL_f}}{e^{-bL_f'}}.$$

Thus, the initially longer cigarette is more effective in reducing the amount of A_t transmitted. However, if we now use basis (B_2), we get

$$\frac{A_t}{L_0 - L_f} = \frac{A_t'}{L_0' - L_f}. \tag{9}$$

In this case, the filtering capacity is independent of the initial length. Since many smokers burn their cigarettes (be they regular or king size) to the same approximate final length before discarding them, basis (B_2) would be indicated. This latter result seems contrary to the claims of certain cigarette advertising.

For those cigarettes having a filter tip of length L_1 and absorption constant b_1, the only term which changes is A_t. In this case we multiply the right-hand side of (6) by $e^{-b_1 L_1}$. If the effective length of the filter tip is L_1' (sometimes part of the filter tip is hollow), then in order that the filter tip be more effective than the tobacco it replaces, we should have $b_1 > bL_1/L_1'$. If our theory is valid, this inequality was apparently not satisfied by many filter-tip cigarettes, since it has been shown experimentally by Consumers Union (*Consumer Reports*, February 1953 and March 1957) that insofar as reducing the nicotine content of the smoke is concerned, there is little to choose between filter-tip or nonfilter-tip cigarettes. However, the average filtered cigarette smoke does contain somewhat less tar than the unfiltered smoke. A direct illustration of this is given by a filter-tip cigarette first put out in 1952. This cigarette did a creditable job of filtering out a high percentage of both nicotine and tar from the tobacco smoke. By 1955 the filter had been more looosely packed for easier smoking, but at the expense of increasing the nicotine and tar levels approximately fourfold and sixfold, respectively, and thus bringing it back to the average filtering level of regular cigarettes.

It is unfortunate that the Consumers Union experiments did not include a determination of $C(x, L)$ after a cigarette had been smoked down. This could then have been used to check the validity of conditions (1), (2), (3), assumed here.

To approximate more closely to actual smoking conditions, we shall consider a cigarette being smoked in the following manner. The cigarette is first smoked a length ΔL_s under steady inhalation and then is allowed to burn a length ΔL_b without inhalation. This process is repeated until the cigarette is discarded.

We now have to solve the differential equation (5) subject to the nonconstant initial condition $C(x, L_0) = C(x)$, where $C(x)$ is a specified function. To solve (5) under this condition, we let

$$C(x, L) = e^{-bx}[R(x, L) + S(L)].$$

Then (5) becomes

$$\frac{\partial R(x, L)}{\partial x} + \frac{\partial R(x, L)}{\partial L} + S'(L) = b[R(x, L) + S(L)] - ab[R(0, L) + S(L)].$$

If we let

$$\frac{\partial R(x, L)}{\partial x} + \frac{\partial R(x, L)}{\partial L} = bR(x, L), \tag{10}$$

subject to the condition

$$R(x, L_0) = e^{bx}C(x), \tag{11}$$

then $S(L)$ must satisfy

$$\frac{dS(L)}{dL} - b(1 - a)S(L) = - abR(0, L), \tag{12}$$

subject to the initial condition $S(L_0) = 0$.

The subsidiary equations for solving (10) are $dx = dL = dR(x, L)/b$. Consequently, $R(x, L) = e^{bx}\psi(L - x)$, where $\psi(x)$ is an arbitrary function of x. To determine $\psi(x)$, we use (11). Thus we find that $R(x, L) = e^{bx}C(L_0 - L + x)$.

We can now solve (12), and the solution is

$$S(L) = abe^{b(1-a)L} \int_L^{L_0} e^{-b(1-a)L}C(L_0 - L)dL,$$

whence $C(x, L) = C(L_0 - L + x) + abe^{b(1-a)L - bx} \int_L^{L_0} e^{-b(1-a)L}C(L_0 - L)dL$.

If we now smoke a length ΔL_a under steady inhalation, the concentration function will be given by

$$C_1(x, L_1) = C_0 + abe^{-bx+b(1-a)L_1} \int_{L_1}^{L_0} e^{-b(1-a)L}C_0 dL,$$

where $L_1 = L_0 - \Delta L_a$. After burning a length ΔL_b, the concentration function will then be $C_1^*(x, L_2) = C_1(x + \Delta L_b, L_1)$. Here we have $L_2 = L_1 - \Delta L_b = L_0 - \Delta L_a - \Delta L_b$. On repeating another cycle we get

$$C_2(x, L_3) = C_1^*(L_2 - L_3 + x, L_2) + abe^{-bx+b(1-a)L_3} \int_{L_3}^{L_2} e^{-b(1-a)L}C_1^*(L_2 - L, L_2)dL,$$

$$C_2^*(x, L_4) = C_2(x + \Delta L_b, L_3).$$

By induction, we find that after N cycles,

$$C_N(x, L_{2N-1}) = C_{N-1}^*(L_{2N-2} - L_{2N-1} + x, L_{2N-2}) + abe^{-bx+b(1-a)L_{2N-1}} \int_{L_{2N-1}}^{L_{2N-2}} e^{-b(1-a)L}C_{N-1}^*(L_{2N-2} - L, L_{2N-2})dL,$$

$$C_N^*(x, L_{2N}) = C_N(x + \Delta L_b, L_{2N-1}),$$

where

$$L_{2N-1} = L_0 - N\Delta L_a - (N - 1)\Delta L_b, \qquad L_{2N} = L_0 - N\Delta L_a - N\Delta L_b.$$

The amount A_{t_N} transmitted to the smoker during the Nth cycle is given by

$$A_{t_N} = \int_{L_{2N-1}}^{L_{2N-2}} aC_{N-1}(0, L)e^{-bL}dL.$$

Hence the total amount A_T absorbed by the smoker after N cycles is

$$A_T = \sum_{n=1}^{N} A_{t_n} = \sum_{n=1}^{N} \int_{L_{2n-1}}^{L_{2n-2}} aC_{n-1}(0, L)e^{-bL}dL.$$

Again, in order to determine the effect of filtration alone, we set $a=1$. It then can be established by induction that

$$\frac{C_N(x, L)}{C_0} = 1 + b\Delta L_s e^{-bx}\left[\frac{e^{-(N-1)b\Delta L_b} - 1}{1 - e^{b\Delta L_b}} + \frac{L_0 - L}{\Delta L_s} - (N-1)\left(1 + \frac{\Delta L_b}{\Delta L_s}\right)\right],$$

and that also

$$A_T = C_0\Delta L_s e^{b(N\Delta L_s - L_0)}\left[\frac{1 - e^{Nb\Delta L_b}}{1 - e^{b\Delta L_b}}\right].$$

For a comparison of the filtration efficiency (under discontinuous inhalation) of two cigarettes, identical except for length, it is reasonable to keep ΔL_s and ΔL_b fixed. Then, using basis (B_1) we get

$$\frac{A_T}{A'_T} = \frac{e^{-bL_0}}{e^{-bL'_0}},$$

which corresponds to (8), and to which it reduces if $\Delta L_b = 0$. By using basis (B_2) we get

$$\left(\frac{A_T}{N\Delta L_s}\right)\Big/\left(\frac{A'_T}{N'\Delta L_s}\right) = \left(\frac{1 - e^{-Nb\Delta L_b}}{N}\right)\Big/\left(\frac{1 - e^{-N'b\Delta L_b}}{N'}\right).$$

Since $(1-e^{-x})/x$ is a monotonic decreasing function, it follows that under either basis, (B_1) or (B_2), the longer cigarette is a better filter under discontinuous inhalation which more nearly approximates actual smoking conditions than continuous inhalation.

ANOTHER NOTE ON QUASI-IDEMPOTENT MATRICES

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Huff [1] and Marathe [2] have considered quasi-idempotent matrices. Huff calls the matrix A over the complex field K quasi-idempotent, if there exists a polynomial matrix $F(x)$ over $K[x]$ such that $A^r = F(r)$ for all positive integers r. Use of the canonical form of A permits us to decompose A into pieces, study the pieces separately, and then put the pieces back together again. This seems to give a clearer picture of the subject.

The Jordan canonical form of a matrix yields immediately that $A(A-I)^k=0$ for some positive integer k, if and only if A is similar to the direct sum $J_1 \dotplus \cdots \dotplus J_s \dotplus 0$, where J_i is the Jordan form of the companion matrix of $(x-1)^{s_i}=0$, $i=1, \cdots, s$. Thus $J_i - I_{s_i}$ is nilpotent of index s_i, *i.e.* $(J_i-I_{s_i})^{s_i-1} \neq 0$, but $(J_i-I_{s_i})^{s_i}=0$. For notational simplicity let us drop the subscripts and consider the nilpotent matrix $J-I$ of index k. Since $J=I+(J-I)$ and $(J-I)^k=0$, we have by the binomial expansion

$$J^n = I + n(J-I) + \frac{n(n-1)}{2!}(J-I)^2 + \cdots + \frac{n(n-1)\cdots(n-[k-2])}{(k-1)!}(J-I)^{k-1}$$

for all positive integers n. (In fact, the expansion holds for all real numbers n, as a finite expansion certainly converges. We understand that if $n=1/3$, say, the resulting matrix $J^{1/3}$ is a cube root of J. There are, of course, other cube roots of J.) Hence if $F(x)$ is the polynomial matrix,

$$(1) \qquad I + x(J-I) + \cdots + \frac{x(x-1)\cdots(x-[k-2])}{(k-1)!}(J-I)^{k-1},$$

we have that $J^r = F(r)$ for all real numbers r. Thus the matrix J (and hence the matrix A above) is quasi-idempotent.

Conversely if A is quasi-idempotent, let the Jordan canonical form of A be $J_1 \dotplus \cdots \dotplus J_t$. Then J_i is quasi-idempotent for $i=1, \cdots, t$. Say $G_i(x)$ is the polynomial matrix such that $J_i^r = G_i(r)$ for all positive integers r. Now J_i is nilpotent implies that $G_i(x)$ is the 0 matrix. Hence J_i is either 0 or nonsingular.

Let J be one of these nonsingular blocks and $G(x)$ the corresponding polynomial matrix whose elements are polynomials of degree $k-1$, say. If $F(x)$ is as defined in (1) above, then the elements of both $F(x)$ and $G(x)$ are polynomials in x of degree $k-1$ (or less). Since $G(i)=F(i)$ for $i=0, \cdots, k-1$, the polynomial elements of both $G(x)$ and $F(x)$ coincide and so $F(x)=G(x)$. But this implies that $(J-I)^n=0$ for $n>k-1$. Hence J is the Jordan form of the companion matrix of $(x-1)^k=0$.

Thus the necessary and sufficient condition that $A(A-I)^k=0$ for some

positive integer k is also a necessary and sufficient condition for A to be quasi-idempotent. Moreover $A^r = F(r)$ for all positive integers r implies $A^r = F(r)$ for all real numbers r. Note that if A is nonsingular $A^0 = I$. But if A is singular (and quasi-idempotent), then $A = P(A_1 \dot{+} 0)P^{-1}$, where A_1 is nonsingular and quasi-idempotent, and $A^0 = P(I \dot{+} 0)P^{-1}$, where I is the unique identity for A_1.

An idempotent matrix is similar to the direct sum $I \dot{+} 0$. A quasi-idempotent matrix, as noted above, allows the use of connecting 1's in the nonsingular portion of the direct sum. Thus any idempotent matrix of order 3 is similar to one of I, 0, $I_1 \dot{+} 0$, $I_2 \dot{+} 0$. Any quasi-idempotent matrix of order 3 which is not idempotent is similar to one of

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In particular a singular quasi-idempotent matrix A of order 3 which is not idempotent is similar to $J \dot{+} 0$, where

$$J = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Since $(J-I)^2 = 0$, we have $J^n = I + n(J-I)$ for all real numbers n. Thus

$$F(x) = I + x(J - I) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

for the matrix J. If $A = P(J \dot{+} 0)P^{-1}$, then

$$F(x) = P \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1}$$

for the matrix A. If

$$P = \begin{bmatrix} -9 & -12 & -2 \\ -1 & -1 & 0 \\ 3 & 4 & 1 \end{bmatrix}, \quad \text{then} \quad P^{-1} = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 3 & -2 \\ 1 & 0 & 3 \end{bmatrix},$$

and we get the A and $F(x)$ of the example of Huff [1].

References

1. G. B. Huff, On quasi-idempotent matrices, this MONTHLY, vol. 62, 1955, pp. 334–339.
2. C. R. Marathe, A note on quasi-idempotent matrices, this MONTHLY, vol. 63, 1956, pp. 632–635.

SOME RESULTS ON RECURRENT EVENTS*

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The purpose of this paper is to develop and illustrate a fundamental theorem for recurrent events occurring at times t where t is a *continuous* real variable, analogous to the results obtained in [1] for the discrete case.

We shall consider the event E to be recurrent when the following conditions are met:

(a) There exists a rule which determines whether E does or does not occur at any time t.
(b) If E occurs at time t, then it occurs at time $t+\tau$, if and only if it occurs at time τ after the time axis has been shifted so that its origin is at t.
(c) If $u(t)dt$ is the element of probability that E occurs at time t, and $u(t, t+\tau)dtd\tau$ is the element of probability of the joint occurrence of E at times t and $t+\tau$, then $u(t, t+\tau)=u(t)u(\tau)$.

Let us denote by $f(t)dt$ the element of probability that a recurrent event E occurs *for the first time* at time t, the process starting at $t=0$, and let us write the generating functions corresponding to $u(t)$ and $f(t)$ as

$$U(\theta) = \int_0^\infty u(y)e^{\theta y}dy, \tag{1}$$

$$F(\theta) = \int_0^\infty f(x)e^{\theta x}dx. \tag{2}$$

FUNDAMENTAL THEOREM. *If $U(\theta)$ and $F(\theta)$ are defined as in* (1) *and* (2) *above, then*

$$F(\theta) = \frac{U(\theta)}{1 + U(\theta)}.$$

If E is a recurrent event, it can easily be shown that $u(t)$ and $f(t)$ are related as follows,

$$u(t) = f(t) + \int_0^t u(t - x)f(x)dx. \tag{3}$$

Either E occurs for the first time at t or it occurs previously at time $t-x$, where $0 \leqq x \leqq t$. Writing the product of $U(\theta)$ and $F(\theta)$ as a double integral, we get

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$$U(\theta)F(\theta) = \int_0^\infty \int_0^\infty u(y)f(x)e^{\theta(y+x)}dxdy, \tag{4}$$

and letting $t=y+x$ with x fixed, (4) becomes

$$U(\theta)F(\theta) = \int_0^\infty \int_0^t u(t-x)f(x)e^{\theta t}dxdt \tag{5}$$

$$= \int_0^\infty \left[\int_0^t u(t-x)f(x)dx \right] e^{\theta t}dt. \tag{6}$$

Substituting (3) for the quantity in brackets, we get

$$U(\theta)F(\theta) = \int_0^\infty [u(t) - f(t)]e^{\theta t}dt = U(\theta) - F(\theta). \tag{7}$$

Upon solving for $F(\theta)$, we get

$$F(\theta) = \frac{U(\theta)}{1 + U(\theta)}, \tag{8}$$

and this completes the proof of the theorem.

Example 1. Let us consider a Poisson process, for which the probability element of the occurrence of an event A at time t is λdt. (As can easily be checked, the occurrence of A is a recurrent event.) Since $u(t)=\lambda$, we have

$$U(\theta) = \int_0^\infty \lambda e^{\theta t}dt = -\frac{\lambda}{\theta} \quad (\theta < 0),$$

and the fundamental theorem yields $F(\theta)=\lambda/(\lambda-\theta)$. It is readily verified that this is the generating function of $f(t)=\lambda e^{-\lambda t}$, and we, thus, have an alternative way of proving the well-known result that in a Poisson process the waiting times between successive occurrences are exponentially distributed.

Example 2. Let us consider two independent Poisson processes for which the element of probability of the occurrence of events A_1 and A_2 at time t are $\lambda_1 dt$ and $\lambda_2 dt$, respectively. The recurrent event which we shall consider is the event that either A_1 or A_2 occurs, and that with this occurrence A_1 and A_2 will have occurred an equal number of times. (This is analogous in the discrete case to two players being even in a sequence of independent two-person games with constant probability and constant pay-off.)

Using the known result that the probability element for the nth occurrence of A_1 at time t is

$$\frac{\lambda_1^n e^{-\lambda_1 t} t^{n-1}}{(n-1)!}dt,$$

and the probability that A_2 occurs exactly n times on the interval from 0 to t is $\lambda_2^n e^{-\lambda_2 t} t^n/n!$, we have by symmetry

$$u(t) = 2\sum_{n=1}^{\infty} \frac{(\lambda_1\lambda_2)^n e^{-(\lambda_1+\lambda_2)t} t^{2n-1}}{(n-1)!\,n!}. \tag{9}$$

Substitution into (1) yields

$$\begin{aligned} U(\theta) = \int_0^{\infty} u(t)e^{\theta t}dt &= \sum_{n=1}^{\infty}\binom{2n}{n}\left[\frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2-\theta)^2}\right]^n \\ &= \left[1 - \frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2-\theta)^2}\right]^{-1/2} - 1, \end{aligned} \tag{10}$$

and according to the fundamental theorem we finally get

$$F(\theta) = \frac{U(\theta)}{1+U(\theta)} = 1 - \left[1 - \frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2-\theta)^2}\right]^{1/2}. \tag{11}$$

To find $f(t)$, let us first write $F(\theta)$ as

$$F(\theta) = \sum_{n=1}^{\infty} \frac{1}{2n-1}(-1)^n\binom{-\frac{1}{2}}{n}\left[\frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2-\theta)^2}\right]^n, \tag{12}$$

and the individual terms of the summation can readily be identified with the generating functions of gamma distributions. Using this fact we get

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{2n-1} \cdot \frac{(\lambda_1\lambda_2)^n e^{-(\lambda_1+\lambda_2)t} t^{2n-1}}{(n-1)!\,n!}. \tag{13}$$

It is of interest to note that if we write (9) as

$$u(t) = \sum_{n=1}^{\infty} g(n, \lambda_1, \lambda_2, t), \tag{14}$$

then (13) becomes

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{2n-1} g(n, \lambda_1, \lambda_2, t), \tag{15}$$

which parallels the result given in [1] for the discrete case.

If we refer to a recurrent event as *certain* if $\int_0^{\infty} f(t)dt = F(0) = 1$ and as *uncertain* if $\int_0^{\infty} f(t)dt = F(0) < 1$, it can readily be shown in our example that the recurrent event is *certain* if $\lambda_1 = \lambda_2$ and *uncertain* if $\lambda_1 \neq \lambda_2$.

Reference

1. W. Feller, An Introduction to Probability Theory and its Applications, vol. 1, New York, 1950.

NUMERICAL DIFFERENTIATION FORMULAS

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1. Introduction. In a recent paper R. T. Gregory [1] has described an algorithm for finding the coefficients in certain formulas, which express the derivatives of an interpolation polynomial at a set of equally spaced points, as linear combinations of the given functional values. This algorithm involves the solution of systems of linear equations, or equivalently, the inversion of certain Vandermonde matrices.

In the following, we obtain explicit representations for the same coefficients, and in such a form that their use is in no way restricted to the tabular points. These representations are in terms of Stirling numbers, which are well known and extensively tabulated. This in turn enables one to determine directly the matrix inverses mentioned above. The authors expect to present further applications of the above to more general matrices, and to some general interpolation problems.

2. Numerical Differentiation Formulas. Let values y_i be given at points $x_i=x_0+ih$, $i=0, 1, \cdots, n$. (For simplicity of notation we consider n as being fixed throughout.) Further, define the factorial polynomial of degree k by

$$p_k(x) = \sum_{j=1}^{k} S_k^j x^j,$$

where the S_k^j are the Stirling numbers of the first kind (*cf.*, for example, Jordan [2], p. 163). If $y=f(x)$ is the polynomial of degree n passing through the $n+1$ points (x_i, y_i), we have

$$h^k f^{(k)}(x) = h^k \frac{d^k f}{dx^k} = \sum_{j=k}^{n} \frac{1}{j!} \sum_{r=k}^{j} p_k(r) S_j^r m^{r-k} \Delta_{j/2}^{(j)},$$

where m is any real number and $x=x_0+mh$ (Kopal [3], p. 98). Since $\Delta_{j/2}^{(j)}$ is independent of r, we have

$$h^k f^{(k)}(x) = \sum_{j=k}^{n} \frac{\Delta_{j/2}^{(j)}}{j!} C_{mj}^k,$$

with

$$C_{mj}^k = \sum_{r=k}^{j} p_k(r) S_j^r m^{r-k}. \tag{1}$$

Applying the relation $\Delta_{j/2}^{(j)} = \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} y_i$ ([2], p. 9), we obtain

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$$h^k f^{(k)}(x) = \sum_{j=k}^{n} \frac{C_{mj}^k}{j!} \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} y_i,$$

and, finally,

$$h^k f^{(k)}(x) = \sum_{i=0}^{n} A_{mi}^k y_i, \tag{2}$$

where

$$A_{mi}^k = \sum_{j=k}^{n} \frac{(-1)^{i+j} \binom{j}{i} C_{mj}^k}{j!}, \qquad \binom{j}{i} = \begin{cases} 1 \text{ if } i = 0 \\ 0 \text{ if } i > j \end{cases}. \tag{3}$$

If the values of y_i are those of a function $y=y(x)$ at the points $x_i=x_0+ih$, where the function has a continuous derivative of order $n+1$, we have

$$h^k y^{(k)}(x) = \sum_{i=0}^{n} A_{mi}^k y_i + O(h^{n+1}).$$

This equation, along with (3) and (1), enables one to write formulas for the derivatives of $y(x)$, with coefficients of the y_i given explicitly. The derivatives of $y(x)$ at the tabular points may be found from the equation by setting $m = 0, 1, \cdots, n$.

3. A matrix inverse. Even though the determination of the coefficients A_{mi}^k is now complete, it is of interest to invert the Vandermonde matrix

$$M(m) = \begin{pmatrix} 1 & 1 & 1 & \cdot & 1 \\ -m & 1-m & 2-m & \cdot & n-m \\ (-m)^2 & (1-m)^2 & (2-m)^2 & \cdot & (n-m)^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (-m)^n & (1-m)^n & (2-m)^n & \cdot & (n-m)^n \end{pmatrix},$$

which appears in Gregory's algorithm. In fact, the algorithm can be extended and written in the form

$$\begin{pmatrix} A_{m0}^0 & A_{m0}^1 & \cdots & A_{m0}^n \\ A_{m1}^0 & A_{m1}^1 & \cdots & A_{m1}^n \\ \cdot & \cdot & \cdot & \cdot \\ A_{mk}^0 & A_{mk}^1 & \cdots & A_{mk}^n \\ \cdot & \cdot & \cdot & \cdot \\ A_{mn}^0 & A_{mn}^1 & \cdots & A_{mn}^n \end{pmatrix} = M^{-1}(m) \begin{pmatrix} 1 & 0 & 0 & \cdot & 0 & \cdot & 0 \\ 0 & 1! & 0 & \cdot & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & k! & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & \cdot & n! \end{pmatrix}.$$

If we now write $M^{-1}(m) = \{a^m_{\lambda\mu}\}$; $\lambda, \mu = 1, 2, \cdots, n+1$, we have immediately

$$a^m_{\lambda\mu} = \frac{A^{\mu-1}_{m,\lambda-1}}{(\mu-1)!},$$

where the $A^{\mu-1}_{m,\lambda-1}$ is given by (3) and (1). Computationally, the easily obtained identity

$$a^m_{\lambda,\mu} = (-1)^{\mu+1}\, a^{n-m}_{n+2-\lambda,\mu}$$

may be of interest. (For the special case where m is an integer, *cf.* [3], p. 91.)

References

1. R. T. Gregory, A method for deriving numerical differentiation formulas, this MONTHLY, vol. 64, 1957, pp. 79–82.
2. Charles Jordan, Calculus of Finite Differences, 2nd ed., New York, 1950.
3. Zdenek Kopal, Numerical Analysis, New York, 1955.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE PRODUCT OF CERTAIN POLYNOMIALS ANALOGOUS TO THE HERMITE POLYNOMIALS

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The Hermite polynomial $H_m(x)$ defined by $e^{2xt-t^2} = \sum_{m=0}^{\infty} H_m(x)t^m/m!$ satisfies the identities

$$H_m(x)H_n(x) = \sum_{r=0}^{\min(m,n)} 2^r r! \binom{m}{r}\binom{n}{r} H_{m+n-2r}(x), \tag{1}$$

$$H_{m+n}(x) = \sum_{r=0}^{\min(m,n)} (-1)^r 2^r r! \binom{m}{r}\binom{n}{r} H_{m-r}(x)H_{n-r}(x). \tag{2}$$

These formulas are due to Nielsen [5, pp. 31–33]; (1) was rediscovered by Dhar [1] and Feldheim [4].

If we put

$$e^{3xt-t^3} = \sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} \tag{3}$$

(compare [3], p. 268, formula (4)), then

$$\begin{aligned}\sum_{m=0}^{\infty} \phi_m(x) \frac{t^m}{m!} \sum_{n=0}^{\infty} \phi_n(x) \frac{z^n}{n!} &= e^{3x(t+z)-(t+z)^3} e^{3tz(t+z)} \\ &= \sum_{h=0}^{\infty} \phi_h(x) \frac{(t+z)^h}{h!} \sum_{k=0}^{\infty} \frac{3^k (tz)^k (t+z)^k}{k!} \\ &= \sum_{h=0}^{\infty} \phi_h(x) \sum_{k=0}^{\infty} 3^k (tz)^k \frac{(t+z)^{h+k}}{h!k!} \\ &= \sum_{k,h=0}^{\infty} 3^k \binom{h+k}{h} \phi_h(x) \sum_{r+s=h+k} \frac{t^{k+r} z^{k+s}}{r!s!}.\end{aligned}$$

Thus $k+r=m$, $k+s=n$, $r+s=h+k$, which yields $r=m-k$, $s=n-k$, $h=m+n-3k$. Hence we get

$$\frac{\phi_m(x)\phi_n(x)}{m!n!} = \sum_{k=0,\,3k\leq m+n}^{\min(m,n)} 3^k \frac{(m+n-2k)!}{k!(m-k)!(n-k)!(m+n-3k)!} \cdot \phi_{m+n-3k}(x),$$

$$\phi_m(x)\phi_n(x) = \sum_{k=0,\,3k\leq m+n}^{\min(m,n)} 3^k k! \binom{m}{k}\binom{n}{k} \frac{(m+n-2k)!}{(m+n-3k)!} \phi_{m+n-3k}(x). \tag{4}$$

For the proof compare Watson [6].

Similarly it follows from

$$\sum_{h=0}^{\infty} \phi_h(x) \frac{(t+z)^h}{h!} = \sum_{m=0}^{\infty} \phi_m(x) \frac{t^m}{m!} \sum_{n=0}^{\infty} \phi_n(x) \frac{z^n}{n!} \cdot \sum_{k=0}^{\infty} (-3)^k \frac{k(tz)^k (t+z)^k}{k!}$$

that

$$\phi_{m+n}(x) = \sum_{r,s} (-3)^{r+s} \binom{m}{r}\binom{n}{s} \frac{(m-r)!(n-s)!}{(m-2r-s)!(n-r-2s)!} \phi_{m-2r-s}(x) \cdot \phi_{n-r-2s}(x), \tag{5}$$

the summation extending over all nonnegative r, s such that $2r+s \leqq m$, $r+2s \leqq n$.

The formulas (4) and (5) may be compared with (1) and (2), respectively.

It is immediate from (3) that

$$\phi_m(x) = \sum_{3r\leq m} (-1)^r \frac{m!}{r!(m-3r)!} (3x)^{m-3r}. \tag{6}$$

Substituting from (6) in (4), we find that the latter is equivalent to

$$(7) \qquad \sum_s \binom{v}{s}\binom{m+n-3v}{m-3s} = \sum_r (-3)^r \binom{v}{r}\binom{m+n-2r}{m-r}.$$

But as was pointed out by Drazin [2], (7) is an immediate consequence of the relation

$$(1+x^3)^v(1+x)^{m+n-3v} = \left\{1 - \frac{3x}{(1+x)^2}\right\}^v (1+x)^{m+n}.$$

Thus we have another proof of (4).

References

1. S. C. Dhar, On the product of parabolic cylinder functions, Bull. Calcutta Math. Soc., vol. 26, 1934, pp. 57–64.

2. M. P. Drazin, Another note on Hermite polynomials, this MONTHLY, vol. 64, 1957, pp. 89–91.

3. A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, Higher Transcendental Functions, vol. 3, New York, 1955.

4. E. Feldheim, Quelques nouvelles relations pour les polynomes d'Hermite, J. London Math. Soc., vol. 13, 1938, pp. 22–29.

5. N. Nielsen, Recherches sur les polynomes d'Hermite, Danske Vid. Sels., Math.-Fys. Medd., I, 6, pp. 1–78.

6. G. N. Watson, A note on the polynomials of Hermite and Laguerre, J. London Math. Soc., vol. 13, 1938, pp. 29–32.

A PROOF OF A THEOREM ON JACOBIANS*

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1. Let x, y denote points in a d-dimensional Euclidean space E^d, and let F be a C^1-mapping from E^d into another d-dimensional space.

We denote by $x^1, \cdots, x^d$, the coordinates of x and by $f^1(x), \cdots, f^d(x)$ those of $F(x)$. It is well known that if the Jacobian $\det. J(z) = \det. (\partial f^i(x)/\partial x^j)_{x=z} \neq 0$, *then $f(x)$ is a one-to-one and open mapping around z. The purpose of this note is to give a simple proof of this theorem.*

2. **Proof of the theorem.** We may assume without loss of generality that $z=0$. We shall use the vector and matrix notations, and shall consider matrices as linear operators on this vector space. Since $J(x)$ is continuous in x, there exists a small positive ϵ such that if $|x| = \mathrm{Max}_i |x^i| < \epsilon$ then for any vector u

$$(1) \qquad |J(x)u - J(0)u| \leqq (1-\epsilon)|J(0)u|.$$

This is possible because $J(0)$ is nonsingular. Now consider a curve $F(x+ty)$ where t ranges over the unit interval $[0, 1]$. We assume that $|x+ty| < \epsilon$ for all t in $[0, 1]$.

* This work was done with the support of the National Science Foundation and the Sloan Foundation.

Define

$$(2) \qquad (\partial/\partial t)F(x+ty) = \lim_{h\to 0} (1/h)(F(x+(t+h)y) - F(x+ty)).$$

Then by easy computations,

$$(3) \qquad (\partial/\partial t)F(x+ty) = J(x+ty)y.$$

Integrate the left hand side from 0 to 1, and we have

$$(4) \qquad F(x+y) - F(x) = \int_0^1 J(x+ty)y\,dt.$$

Now suppose that $F(x+y)=F(x)$. Then

$$\begin{aligned} 0 &= \left| \int_0^1 J(x+ty)y\,dt \right| \\ &= \left| \int_0^1 J(0)y\,dt + \int_0^1 (J(x+ty) - J(0))y\,dt \right| \\ &\geqq |J(0)y| - (1-\epsilon)|J(0)y| = \epsilon |J(0)y|. \end{aligned}$$

Hence $J(0)y=0$, which implies $y=0$ because $J(0)$ is nonsingular.

Next, we assume $F(0)=0$ and prove that the image of $|x|<\epsilon/2$ contains the sphere $|z|<\delta=\epsilon/2 \inf_{|x|=\epsilon/2} |J(0)x|$. Let z_0 be given, $|z_0|<\delta$. Find x_0 such that

$$|F(x_0) - z_0| = \inf_{|x|\leqq\epsilon/2} |F(x) - z_0|.$$

For any x, $|x|=\epsilon/2$, by (4) and (1)

$$|F(x)| = \left| J(0)x + \int_0^1 (J(tx) - J(0))x\,dt \right| \geqq \epsilon |J(0)x| \geqq 2\delta,$$

$$|F(x) - z_0| \geqq \delta > |F(0) - z_0| \geqq |F(x_0) - z_0|.$$

Hence $|x_0|<\epsilon/2$, and y can be chosen so that $|x_0+ty|\leqq\epsilon/2$ for $0\leqq t\leqq 1$, and $J(0)y=-\eta(F(x_0)-z_0)$ for some small positive η. Then, by (4) and (1) $|F(x_0+y) - F(x_0) - J(0)y| \leqq (1-\epsilon)|J(0)y|$, $|F(x_0+y)-z_0| \leqq |F(x_0)-z_0+J(0)y| + (1-\epsilon)|J(0)y| \leqq (1-\eta\epsilon)|F(x_0)-z_0|$. From the definition of x_0 it follows that $F(x_0)-z_0=0$. The theorem is hence proved.

Remark. We can extend this result to a certain more general infinite dimensional space if we can give a proper formulation to the continuity and the nonsingularity of $J(x)$.

CUBIC INVERSION

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The purpose of this paper is to develop inversion, using a cubic as the base curve instead of a conic section. To simplify algebraic manipulations, any cubic of the pencil of syzygetic cubics has been chosen as the base curve rather than the general cubic. This is permissible; for any nonsingular cubic by proper transformations can be put in the form of one of these cubics.

The equations of the inverse transformation. Take a fixed point $A_3(0, 0, 1)$ as origin, and take as the fixed fundamental base curve a cubic of the syzygetic pencil $F \equiv x_1^3+x_2^3+x_3^3-3\lambda x_1x_2x_3=0$. Points collinear with the origin and conjugate with respect to the base curve are said to be inverse. A line through the origin and a general point $P'(x_1', x_2', x_3')$ intersects the polar line of P' with respect to the base cubic in a point $P(x_1, x_2, x_3)$. P and P' are conjugate points.

When the point $P(x_1, x_2, x_3)$, the intersection of A_3P' and the polar line of P', is found, the following cubic inverse transformation is established:

$$x_1 : x_2 : x_3 = f_1 : f_2 : f_3, \tag{1}$$

where

$$f_1 = \lambda x_1'^2 x_2' - x_1' x_3'^2, \qquad f_2 = \lambda x_1' x_2'^2 - x_2' x_3'^2,$$

$$f_3 = x_1'^3 + x_2'^3 - 2\lambda x_1' x_2' x_3'.$$

The cubics $f_1=0, f_2=0, f_3=0$ have seven points in common.* Thus the cubics $\alpha_1f_1+\alpha_2f_2+\alpha_3f_3=0$, where the α_i are general constants, form a net of cubics through these seven base points.

Inverses of points and curves. Any point on the base cubic, F, is its own inverse; thus all points of the base cubic are invariant under this transformation.

Each of the points $A_1(1, 0, 0)$ and $A_2(0, 1, 0)$ inverts into $A_3(0, 0, 1)$. The inverse of A_3 is indeterminate. Using the method of infinitesimals, and taking a point $(\epsilon_1, \epsilon_2, 1+\epsilon_3)$ close to A_3 on the line $x_1+\alpha x_2=0$, where $\alpha=-(\epsilon_1/\epsilon_2)$ we find that points close to A_3 invert into points on $x_3=0$. Any point of $x_1=0$ inverts into some other point of $x_1=0$, and likewise the points of $x_2=0$ invert into other points of $x_2=0$, but no point of $x_3=0$ inverts into a point of $x_3=0$.

The inverse of a line through the fixed origin A_3 is a degenerate cubic through all the vertices of the fundamental triangle $A_1A_2A_3$, and it consists of the original line and the conic $\lambda x_1x_2-x_3^2=0$.

The inverse of a line through A_1 or A_2 is a nonreducible cubic through A_3, but not through A_1 or A_2.

The inverse of a general line is a cubic which passes through A_3 but not through A_1 or A_2.

* A certain cubic transformation, this MONTHLY, vol. 59, 1952, p. 314.

The inverse of a general conic is a sextic through the seven base points of the net of cubics $\alpha_1 f_1+\alpha_2 f_2+\alpha_3 f_3=0$, and it has at least a double point at each of these base points.

The inverse of a general n-ic is a $3n$-ic through the seven base points. If the n-ic is of order higher than the first, the base points are at least double points of the $3n$-ic.

Every n-ic through A_3 is transformed into a degenerate $3n$-ic. If A_3 is a k-ple point of the n-ic, then the inverse curve contains the conic $\lambda x_1x_2-x_3^2=0$ to the k-th power.

The reader will recall that under quadric inversion, the inverse of each curve through A_3 contains a line as part of it. Thus in cubic inversion we have a similar situation.

The effects of the transformation on the multiple points of an n-ic. An n-ic which has a k-ple point at A_1 but does not go through A_2 or A_3 is transformed into a $3n$-ic which does not go through A_1 or A_2, but has an n-ple point at A_3.

Since $k\leqq n$, the order of the singularity is never decreased. If the n-ic is reducible and $k=n$, the order of the singularity is the same. While if $k<n$, the order is increased.

An n-ic which has a k-ple point at A_2, but does not go through A_1 or A_3, is transformed into a $3n$-ic which does not go through A_1 or A_2, but has an n-ple point at A_3.

An n-ic which does not pass through A_1 or A_2, but has an s-ple point at A_3, is transformed into a $3n$-ic with s-ple points at both A_1 and A_2 and an n-ple point at A_3, $(s\leqq n)$.

An n-ic with a k-ple point at A_1, an r-ple point at A_2, and an s-ple point at A_3 is transformed into a $3n$-ic with an n-ple point A_3 and s-ple points at both A_1 and A_2.

Thus an s-ple point at A_3 can reduce a singularity at A_1 and A_2, but a singularity at A_3 is never reduced.

Some examples of the effects of the cubic inverse transformation are:

(a) A cubic with a double point at A_1 or A_2 is transformed into a nonic with a triple point at A_3.
(b) A cubic with a double point at A_3 is transformed into a nonic with double points at A_1 and A_2, and a triple point at A_3.
(c) A conic through A_1 or A_2 is transformed into a sextic with a double point at A_3.
(d) A conic through A_3 is transformed into sextic through A_1 and A_2 with a double point at A_3.

A general n-ic of order higher than the first which has no singularities is transformed into a $3n$-ic with at least a double point at each of the seven base points.

Invariants. There are $3n$ invariant points on any n-ic. These are the points

in which it intersects the base cubic since all points of the base cubic are invariant.

Any cubic G_3 intersects the base cubic in nine invariant points. Three of these points lie on the straight line, of which G_3 is the inverse. By Noether's Law the other six points of intersection lie on a conic.

In general the cross ratio of four collinear points under this transformation is not invariant, but the cross ratio on the lines $x_1=0$ and $x_2=0$ is a covariant.

In general, distance is not invariant but is a covariant on the lines $x_1=0$ and $x_2=0$.

The reader will recall that in quadric inversion the inverse points P and P', and the fixed origin A_3 are collinear, and are so related that the product $A_3P \cdot A_3P'=k^2$, where the base curve is a circle with center A_3 and radius k. Thus the product $A_3P \cdot A_3P'$ is always constant.

In cubic inversion, the product

$$A_3P \cdot A_3P' = \frac{(x^2+y^2)(\lambda xy-1)}{x^3+y^3-2\lambda xy}$$

is a variable quantity. This is easily understood since each point P' has a different polar conic.

If $A_3P \cdot A_3P'=\lambda$, the cubic inverse transformation reduces to the quadratic transformation $x_1:x_2:x_3=\lambda x_1':\lambda x_2':x_1'^2+x_2'^2$. This is equivalent to quadric inversion with respect to a circle with center at A_3 and $r^2=\lambda$.

References

1. Surendramohan Ganguli, The Theory of Plane Curves, University of Calcutta, 1925.
2. William C. Graustein, Introduction to Higher Geometry, New York, 1930.
3. Harold Hilton, Plane Algebraic Curves, Oxford University, 1920.
4. Frank Morley and F. V. Morley, Inversive Geometry, New York, 1933.
5. A. B. Basset, Cubic and Quartic curves, Cambridge, 1901.
6. Heinrich Schroeter, Theorie der Ebenen Kurven, Leipzig, 1888.
7. Julian Lowell Coolidge, A Treatise on Algebraic Plane Curves, Oxford, 1931.
8. Arnold Emch, Lecture Notes on Algebraic Geometry, unpublished.

THE PROBABILITY OF A SADDLEPOINT

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A *saddlepoint* of a matrix (A_{ij}) is a particular pair (r, s) such that A_{rs} is the minimum element in the rth row and the maximum element in the sth column. (See Chapter I of [1] for the importance of this concept in the Theory of Games.) The object of this note is to make precise the statement (intuitively plausible and confirmed by experience) that "large matrices rarely have saddlepoints."

THEOREM. *Let the elements of the $m \times n$ matrix (A_{ij}) be independent random variables with the same continuous distribution function. Then the probability that (A_{ij}) has a saddle-point is $P(m, n)=m!n!/(m+n-1)!$*

Proof. Consider the events

$$E: (A_{ij}) \text{ has a saddlepoint.}$$

$$E(r, s): (A_{ij}) \text{ has } (r, s) \text{ as saddlepoint,}$$

$$F: (A_{ij}) \text{ has all its elements distinct.}$$

Our continuity assumption ensures $\Pr(F) = 1$, so that

$$P(m, n) = \Pr(E) = \Pr(E \cap F).$$

It is easy to show that if (r, s) and (t, u) are both saddle-points, then $A_{rs} = A_{tu}$. Thus $E \cap F$ is the *disjoint* union of the events $E(r, s) \cap F$, and so

$$P(m, n) = \sum_{r,s} \Pr(E(r, s) \cap F).$$

By symmetry (we omit the formal argument), $\Pr(E(r, s) \cap F)$ has the same value for all pairs (r, s), and so

$$P(m, n) = mn \Pr(E(1, 1) \cap F).$$

The evaluation of $\Pr(E(1, 1) \cap F)$ is a matter of elementary combinatorial probability; the $(m+n-1)!$ possible orderings of the elements of the union of the first row and first column of (A_{ij}) are clearly equiprobable (again we omit the formalization) and precisely $(m-1)!(n-1)!$ of them make $(1, 1)$ a saddlepoint, so that

$$\Pr(E(1, 1) \cap F) = (m - 1)!(n - 1)!/(m + n - 1)!$$

and the proof is complete.

Note that $P(m, n)$ does not depend on the distribution function. This is *not* the case for discontinuous distributions, however. For a nontrivial example, consider a 2×2 matrix (A_{ij}) of independent random variables with the same Bernoulli distribution

$$\Pr(A_{ij} = 0) = p, \qquad \Pr(A_{ij} = 1) = q = 1 - p;$$

we find by enumerating the 16 possible numerical matrices (each with the appropriate probability) that the probability of a saddlepoint is $1 - 2p^2q^2$, which of course depends on p.

Reference

1. J. C. C. McKinsey, Introduction to the Theory of Games, New York, 1952.

NOTE ON VERTICES IN EUCLIDEAN 3-SPACE

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The object of this note is to indicate by theorem and example that, strangely enough, a point of a curve where the radius of the osculating sphere has an extreme value need not be the same as a point where the osculating sphere has at least five consecutive points in common with a given curve. Throughout, the notation and results of *Lectures on Classical Differential Geometry* by D. J. Struik are used.

The following relationships are well known:

$$K^2 = \vec{x}'' \cdot \vec{x}'', \qquad \tau = (\vec{x}'\vec{x}''\vec{x}''')/K^2; \tag{1}$$

$$\text{Frenet formulae } \vec{t}' = K\vec{n}, \qquad \vec{n}' = -K\vec{t} + \tau\vec{b}, \qquad \vec{b}' = -\tau\vec{n}; \tag{2}$$

$$r = \text{radius of osculating sphere} = [R^2 + (TR')^2]^{1/2}, \qquad T = 1/\tau, \qquad R = 1/K, \qquad R' = -K'/K^2 \tag{3}$$

DEFINITION 1. *A space curve* $C: \vec{x} = [x^1(s), x^2(s), x^3(s)]$ *is called a curve of type* I *if and only if*

(a) $x^i(s)$ *is analytic for each* $i = 1, 2, 3$;
(b) *the parameter* s *represents arc length*;
(c) *neither* $K = 0$ *nor* $\tau = 0$ *at any point of* C;
(d) *the curve is not spherical.*

From (b) and (c), neither $\vec{x}'$ nor $\vec{x}' \times \vec{x}''$ is zero at any point of a curve of type I, since $\vec{x}' = \vec{t}$ and $\vec{x}' \times \vec{x}'' = K(\vec{t} \times \vec{n})$. Also, since $\vec{x}$ is analytic, it follows from equation (1) that K is finite. Hence, since $K \neq 0$, by (c), it follows readily from equation (1) that K' is also finite. Since $\tau \neq 0$, it follows from equation (3) that for every curve of type I

$$0 < r = [R^2 + (TR')^2]^{1/2} < \infty. \tag{4}$$

Throughout, the letter C represents a curve of type I, the letter Σ represents a given sphere, and P represents the point corresponding to parameter value s_0.

DEFINITION 2. *Let* C *and* Σ *meet in point* P. *Let* A *be a point on* C, *near* P, *and let* AD *be its distance to* Σ. *Then* Σ *has a contact of order* n *with* C *if and only if*

$$\lim_{A \to P \text{ on } C} \frac{AD}{(AP)^k}$$

is finite and not zero for $k = n+1$, *but is zero for* $k = n$.

Let C be given by $\vec{x} = x(s)$, and let Σ be given by $f(x^1, x^2, x^3) = (\vec{x} - \vec{a}) \cdot (\vec{x} - \vec{a})$

* The author is now at Purdue University.

$-d^2=0$. Let $f(s)=(\vec{x}(s)-\vec{a})\cdot(\vec{x}(s)-\vec{a})-d^2$. Then the following is known (Struik, p 24):

THEOREM 1. *Necessary and sufficient conditions that Σ have contact of order n at P with C are that $f(s)=f'(s)=\cdots=f^n(s)=0$; $f^{(n+1)}(s)\neq 0$, all at $s=s_0$.*

DEFINITION 3. *A set of $n+1$ points is called spherical if and only if they lie on a sphere.*

DEFINITION 4. *If in any neighborhood of P on C there are $n+1$ spherical points, then the osculating sphere to C at P is said to have at least $n+1$ consecutive points in common with C at P.*

THEOREM 2. *A necessary condition that C and its osculating sphere Σ have at least $n+1$ consecutive points in common at P is that Σ have contact of order at least n with C at P. (This follows analogously as in Struik, p. 10 and p. 24.)*

DEFINITION 5. *A point P of C is defined to be a vertex if and only if the osculating sphere at P has at least five consecutive points in common with C.*

THEOREM 3. *A necessary condition that P be a vertex of C is that, at P*

$$-K\tau K''+2K'^2\tau+KK'\tau'+K^2\tau^3=0.$$

Proof. Let Σ represent the osculating sphere to C at P. By Theorem 1, Theorem 2, and the Frenet formulae, for P to be a vertex, it is necessary that, at $s=s_0$

(a) $$f'(s)=2(\vec{x}-\vec{a})\cdot\vec{t}=0$$

(b) $$f''(s)=2[(\vec{x}-\vec{a})\cdot K\vec{n}+1]=0$$

(c) $$f'''(s)=2(\vec{x}-\vec{a})\cdot(K'\vec{n}+K\tau\vec{b}-K^2\vec{t})=0$$

(d) $$\begin{aligned} f''''(s)=&-2K^2+2(\vec{x}-\vec{a})\cdot[K''\vec{n}+K'(-K\vec{t}+\tau\vec{b})+K'\tau\vec{b}+K\tau'\vec{b}\\ &+K\tau(-\tau\vec{n})-2KK'\vec{t}-K^2K\vec{n}]=0\end{aligned}$$

Since Σ is the osculating sphere, (a), (b), and (c) are satisfied, since

(a_1) $$(\vec{x}-\vec{a})\cdot\vec{t}=0$$

(b_1) $$(\vec{x}-\vec{a})\cdot\vec{n}=-R=-1/K$$

(c_1) $$(\vec{x}-\vec{a})\cdot\vec{b}=RK'/(K\tau)=-R'T=K'/(K^2\tau)$$ (See Struik, p. 25)

Hence, it is necessary that $f''''(s)=0$. Using then (a_1), (b_1), (c_1), equation (d) becomes

(d_1) $$f''''(s)/2=-K''/K+2K'^2/K^2+\tau'K'/(K\tau)+\tau^2=0$$

But since C is a curve of type I, $K\neq 0$, $\tau\neq 0$, so that (d_1) is equivalent to

(5) $$-K''K\tau+2K'^2\tau+KK'\tau'+K^2\tau^3=0.$$

DEFINITION 6. *A stationary point Q of C is a point where $r'=0$, where r is given by* (3).

THEOREM 4. *Every vertex point P of C is a stationary point.*

Proof.

$$r' = [RR' + TR'(T'R' + TR'')]/[R^2 + (TR')^2]^{1/2}$$

By equation (4), it follows then that a necessary and sufficient condition that $r'=0$ is that

$$RR' + TR'(T'R' + TR'') = 0. \tag{6}$$

Since $R=1/K$, $R'=K'/K^2$, $R''=-K''/K^2+2K'^2/K^3$, $T=1/\tau$, $T'=-\tau'/\tau^2$, substitution into (6) and use of Definition 1, part (c), yields the equivalent condition

$$K'[-K\tau K'' + 2K'^2\tau + KK'\tau' + K^2\tau^3] = 0 \tag{7}$$

However, if P is a vertex, by Theorem 3, equation (7) is satisfied. Hence $r'=0$ at P, so that P is a stationary point.

It is now of interest to note that every stationary point need not be a vertex. This is motivated from (7). Since the expression in brackets is necessary for a point to be a vertex, one can still make $r'=0$ by making $K'=0$. The following example results from such reasoning.

Consider a helix which is given by the natural equations $K=\tau=1+s^2$. First it is shown that the conditions of Definition 1 are satisfied.

(a) Consideration of the system of linear differential equations, $\alpha'=K\beta$, $\beta'=-K\alpha+\tau\lambda$, $\lambda'=-\tau\beta$, where in this case K and τ are analytic, yields readily, by use of the argument of Struik, pp. 29–31, the fact that the helix is analytic.

(b) By assumption, s represents arc length.

(c) This condition is obviously satisfied.

(d) A necessary and sufficient condition that a curve be spherical is that $R^2+(TR')^2$ be identically constant (see Struik, p. 32). For the given helix

$$r^2 = R^2 + (TR')^2 \equiv (1 + 8s^2 + 6s^4 + 4s^6 + s^8)/(1 + s^2)^4. \tag{8}$$

Since this last expression is not identically constant, all conditions are satisfied and the helix is a curve of type I.

Moreover, simple computation yields

$$r = (1 + 8s^2 + 6s^4 + 4s^6 + s^8)^{1/2}/(1 + s^2)^2 \tag{9}$$

$$r = 1, \quad r' = 0, \quad r'' = 2, \text{ at } s = 0. \tag{10}$$

Hence, at $s=0$, r not only attains a stationary value, but actually attains an extreme value. However, at $s=0$, the point is not a vertex, since $K=\tau=1+s^2$, $K'=\tau'=2s$, $K''=\tau''=2$, so that equation (5) is not satisfied since $-2+0+0+1=-1\neq 0$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

TRIGONOMETRY ABSTRACTLY TREATED

W. W. SAWYER, University of Illinois

Trigonometry when taught to thirteen year olds is (rightly) treated as a branch of physics. If a pupil asks, "What is an angle of 40°?" the teacher points to this angle on a protractor, that is, explains the idea by means of a physical object. Is it possible to make the ideas of angle, sine, cosine precise without any appeal to figures?

With calculus known, one can of course define arc sin k as $\int_0^k (1-t^2)^{-1/2}dt$, and derive its properties. But at school we form an idea of "angle" before we have met calculus, limits, or any infinite process, even before we have met radian measure. It should be possible to explain angle by *algebraic* reasoning.

The natural way to explain angles would be by stepping off equal intervals around the arc of a circle. If $A, B, C, D, E \cdots$ are evenly spaced points on the circumference of a circle with center O, we should recognize statements such as $\angle AOB = \angle BOC$, $\angle AOC = 2\angle AOB$. In marking out the points A, B, C, D, E, we use compasses to make equal not the *arcs* AB, BC, CD, DE but the direct distances AB, BC, CD, DE. In drawing the circle $ABCDE$ with center O we also use distance.

The starting point is therefore the idea of *distance between points*. This we can handle quite well by means of coordinate geometry, so we may begin with the axioms, or definitions—

(1) *A point is a pair of real numbers* (x, y).

(2) *The distance* s *between two points* (a, b) *and* (c, d) *is a positive (or zero) number such that* $s^2 = (a-c)^2 + (b-d)^2$.

(3) *If we have two sets of points,* $(x_1, y_1) \cdots (x_n, y_n)$ *and* $(X_1, Y_1) \cdots (X_n, Y_n)$, *such that the distance between a pair of points in the first set equals the distance between the corresponding points in the second set, the sets form congruent figures.*

(4) *We call the operation of replacing points by the points of a congruent set a rigid displacement.*

We want to show that there are rigid displacements in which every point of space goes to a corresponding point. We first investigate what conditions such displacements must satisfy; we are particularly interested in rotations, which we must define.

(5) *A movement about a point* (h, k) *means a rigid displacement in which* (h, k) *goes to itself.*

We do not straight away call this a rotation, since it might be for example the reflection; (x, y) goes to $(x, -y)$. This is a movement about $(0, 0)$,since $(0, 0)$ goes to $(0, 0)$ and distances are preserved. We must find some way of distinguishing rotations from reflections.

Consider any movement about the origin $(0, 0)$. Let (x_1, y_1) go to (X_1, Y_1) and (x_2, y_2) to (X_2, Y_2). Then

$$\text{(I)} \qquad x_1^2 + y_1^2 = X_1^2 + Y_1^2$$

$$\text{(II)} \qquad x_2^2 + y_2^2 = X_2^2 + Y_2^2$$

$$\text{(III)} \qquad (x_1 - x_2)^2 + (y_1 - y_2)^2 = (X_1 - X_2)^2 + (Y_1 - Y_2)^2$$

since distances are preserved. Adding (II) to (I) and subtracting (III), we find after removing a factor 2, $x_1x_2+y_1y_2=X_1X_2+Y_1Y_2$.

Hence

THEOREM 1. *Any movement about the origin leaves $x_1x_2+y_1y_2$ unaltered in value.*

(One may comment on the significance of this as scalar product, the work done by a force in a displacement, and so on.)

There is a well known algebraic identity $(x_1^2+y_1^2)(x_2^2+y_2^2)-(x_1x_2+y_1y_2)^2 = (x_1y_2-x_2y_1)^2$. As the left-hand side is unaltered in any movement about the origin, the right-hand side also must be. Accordingly we have

THEOREM 2. *$x_1y_2-x_2y_1$ is either unchanged, or changes only in sign in any movement about the origin.*

There are now three possibilities.

(i) In the movement, for all values of x_1, x_2, y_1, y_2, the expression $x_1y_2-x_2y_1$ is unchanged.

(ii) In the movement, this expression changes its sign for all values of the variables.

(iii) There are some values for which the expression is unchanged, and some for which it changes sign.

(6) A movement of type (i) is called a *rotation*; a movement of type (ii) is called a *reflection*; a movement of type (iii) does not exist, as will be shown later.

Formula for rotation. Let $(1, 0)$ go to (p, q), in a rotation about the origin. Then, since distance is preserved, $p^2+q^2=1$. Let (x, y) go to (X, Y). Since $x_1x_2+y_1y_2$ is unchanged in any motion, and $x_1y_2-x_2y_1$ is unchanged for every point in a rotation, we have

$$\text{(A)} \qquad \begin{aligned} Xp + Yq &= x\cdot 1 + y\cdot 0 = x, \\ -Xq + Yp &= -x\cdot 0 + y\cdot 1 = y. \end{aligned}$$

Solving for X, Y and using $p^2+q^2=1$ we have

$$X = px - qy,\ Y = qx + py. \tag{B}$$

It is easily verified that this formula makes all distances invariant. Hence, if $p^2+q^2=1$, there is one and only one rotation that sends $(1, 0)$ to (p, q).

Formula for reflection. Let $(1, 0)$ go to (p, q) where $p^2+q^2=1$, and (x, y) to (X, Y). For a reflection, $x_1x_2+y_1y_2$ remains the same, while $x_1y_2-x_2y_1$ changes sign. We have

$$Xp + Yq = x,\ -Xq + Yp = -y; \tag{C}$$

whence

$$X = px + qy,\ \ Y = qx - py. \tag{D}$$

Thus there is one and only one reflection that sends $(1, 0)$ to (p, q), while leaving the origin fixed.

Impossibility of mixing rotations and reflections. Possibility (iii) envisages $x_1y_2-x_2y_1$ being unchanged for some values of the variables, while changing sign for others. It is easily verified that, for equations (B), the value is unchanged for all x_1, y_1, x_2, y_2 while for equations (D), the expression always changes sign. A third possibility could only arise if some points obeyed equations (B) while others obeyed equations (D). Let (x_1, y_1) obey (B) and (x_2, y_2) obey (D). A simple calculation shows $X_1X_2+Y_1Y_2=x_1x_2-y_1y_2$. This result can only be reconciled with Theorem (1) if $y_1y_2=0$, that is, if one of the points lies on $y=0$. But when $y=0$, equations (B) and (D) give the same values of (X, Y). Thus there is no possibility of a genuine mixing of the equations. Thus

THEOREM 3. *The only possible types of motion about the origin are the rotations, given by equations* (B), *and the reflections, given by equations* (D).

Rotations and reflections are linear operations; they preserve relations such as $x_1=kx_2$, $y_1=ky_2$ or $x_1+x_2=x_3$, $y_1+y_2=y_3$. This is evident from formulae (B) and (D).

The set of points (ka, kb), where a, b are fixed and k takes all positive values, form a *ray* through the origin. Rotations and reflections transform rays into rays.

Combination of rotations and reflections. Since a rotation leaves all distances, the origin, and all expressions $x_1y_2-x_2y_1$ unaltered, it is evident that a rotation followed by a rotation gives a rotation.

Since reflections change the sign of $x_1y_2-x_2y_1$ it is evident that two successive reflections produce a rotation.

Angles. The measurement of angle is not really investigated in elementary trigonometry except for rational ratios of angles, in effect when $P=n\theta$, $Q=m\theta$,

where m, n are integers. Hence, if the operation given by equations (B) is called rotation through θ, the rotation through $n\theta$ will be the result of carrying out this operation n times.

Rotation through $-\theta$ is the operation that undoes the effect of rotation through θ. The equations for this operation can be written down by inspecting equation (A).

DEFINITION. *The value of p that occurs in equations* (B) *for rotation through θ is called the cosine of θ. The value of q that occurs in equations* (B) *is called the sine of θ.*

By comparing equations (B) and (A), one can see that $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$.

Since $p^2+q^2=1$, $\cos^2\theta+\sin^2\theta=1$.

Rotation through θ thus has the equations

$$X = x\cos\theta - y\sin\theta, \qquad Y = x\sin\theta + y\cos\theta.$$

Sine and cosine of sum. Rotation through $\theta+\phi$ is understood to mean the effect of rotation through θ followed by rotation through ϕ. It can be verified that $\phi+\theta$ gives the same result.

Let rotation through θ send (x_1, y_1) to (x_2, y_2), and rotation through ϕ send (x_2, y_2) to (x_3, y_3).

The equations for this can be written down, and x_3, y_3 found from these equations in terms of x_1, y_1. The resulting equations are, of course, of type (B). The values of p and q can be read off, and give the usual formulas for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$.

THE MECHANICAL BRAIN

W. L. STROTHER, U. S. Naval Ordnance Test Station and University of Miami

Misconceptions concerning high speed computers often arise from newspaper articles suggesting that a computer is some sort of a sensational miracle machine or "mechanical brain." A student may, for example, get such a distorted picture from such an article that he concludes that the computer can do the problems for him, so that there is now no need for him to learn any mathematics. Simply telling him that the machine "can only add, subtract, multiply, and divide" is not sufficient. He has heard, for example, that it can integrate and has no idea how the labor of integration is divided between him and the machine. We can give him a clearer picture by leading him through the coding of one problem than by a plethora of comments about the machine. The following oversimplified example can easily be presented in half of one class period.

Problem. Given the equation $x^3+2x^2+1=0$, take x_0 to be -2 for the first approximation to a root and compute, by Newton's Method, the next n approximations.

Solution for the case $n=1$ without the machine is obtained by replacing x_0 by -2 in the formula $x_1=x_0-f(x_0)/f'(x_0)$ where $f'(x)=3x^2+4x$, getting $x_1=-9/4$.

Solution for this same case by machine: The machine can add, subtract, multiply, divide, "store" information (*i.e.*, write it down somewhere inside the machine), and can print out what it has previously stored. In order to use the machine to solve the problem one must write out its instructions in terms of these operations. (Different machines require different instructions. The following instructions are for a machine which has not yet been built.)

Instructions	Storage Registers		Explanations
I_1 Store -2 in register R_1. Read I_2.	R_1	-2	x
I_2 Store contents of R_1 in R_2. Read I_3.	R_2	-2	x
I_3 Mult. contents of R_1 by contents of R_2, store in R_3. Read I_4.	R_3	4	x^2
I_4 Mult. contents of R_1 by contents of R_3, store in R_4. Read I_5	R_4	-8	x^3
I_5 Store 2 in R_5. Read I_6.	R_5	2	
I_6 Mult. contents of R_5 by contents of R_3, store in R_6. Read I_7.	R_6	8	$2x^2$
I_7 Store 1 in R_7. Read I_8	R_7	1	
I_8 Add contents of R_6 to contents of R_4, store in R_8. Read I_9	R_8	0	x^3+2x^2
I_9 Add contents of R_7 to contents of R_8, store in R_9. Read I_{10}	R_9	1	x^3+2x^2+1
I_{10} Store 3 in R_{10}. Read I_{11}.	R_{10}	3	
I_{11} Store 4 in R_{11}. Read I_{12}.	R_{11}	4	
I_{12} Mult. contents of R_{10} by contents of R_3, store in R_{12}. Read I_{13}.	R_{12}	12	$3x^2$
I_{13} Mult. contents of R_{11} by contents of R_1, store in R_{13}. Read I_{14}.	R_{13}	-8	$4x$
I_{14} Add contents of R_{13} to contents of R_{12}, store in R_{14}. Read I_{15}.	R_{14}	4	$3x^2+4x$
I_{15} Div. contents of R_9 by contents of R_{14}, store in R_{15}. Read I_{16}.	R_{15}	1/4	$f(x)/f'(x)$
I_{16} Store -1 in R_{16}. Read I_{17}.	R_{16}	-1	
I_{17} Mult. contents of R_{15} by contents of R_{16}, store in R_{17}. Read I_{18}.	R_{17}	$-1/4$	$-f(x)/f'(x)$
I_{18} Add contents of R_1 and R_{17} and store in R_1. Read I_{19}.	R_{18}	$-9/4$	$x-f(x)/f'(x)$
I_{19} Print out contents of R_1. Read I_2.			

Note that the first answer $x_1=-9/4$ was obtained by hand calculations with no more time, labor, or knowledge of mathematics than was required to write the instructions for the machine. (Actually the list of instructions above would be longer. Such things as placing the decimals, *etc.*, were intentionally omitted.) But now the machine becomes useful, because instruction 19 starts it back through the entire cycle, and solutions for the cases $n=2, 3, \cdots$ *etc.* are computed with no further writing of instructions until one is ready to stop the machine, and it does this repetitious computation with incomparably greater speed than a man can do it. It can carry out the 19 instructions in the above list a few thousand times per second. Such is the situation whenever one wants the help of the machine. One must understand how to solve the problem in order to instruct the machine how to solve it. The machine will then do the laborious execution of these instructions with much greater speed and accuracy than they can be done by hand.

ANOTHER LOOK AT THE PROBABILITY INTEGRAL

C. P. NICHOLAS, U. S. Military Academy

A well-known method of showing that $\int_0^\infty e^{-x^2}dx=\sqrt{\pi}/2$ is first to demonstrate that the integral is convergent, and then by a transformation of coordinates to set up the equation

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^{\pi/2} \int_0^\infty e^{-\rho^2}\rho d\rho d\theta.$$

From this $[\int_0^\infty e^{-x^2}dx]^2=\pi/4$, and we complete the solution by taking square roots.

An adaptation to instruction at the sophomore level appeared in this MONTHLY, vol. 57, 1950, pp. 412–413. Simplicity was achieved by certain geometric devices, and by the *assumption* that the improper integral is convergent.

The feeling of uneasiness that goes with this sweeping assumption can be avoided by the evaluation offered below. It is believed easy enough for a first course, and the final step establishes both the existence and the value of the integral.

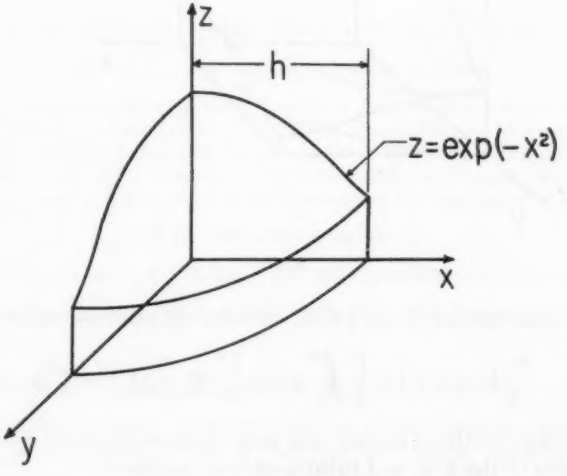

FIG. 1

Consider the surface generated by revolving about the Z-axis the curve $z=e^{-x^2}$, and let V_1 be the volume (Fig. 1) in the first octant bounded by this surface, the coordinate planes, and the cylinder $x^2+y^2=h^2$.

By the method of cylindrical shells

$$V_1 = \int_0^h \frac{\pi}{2} xe^{-x^2}dx = \frac{\pi}{4}[1 - e^{-h^2}].$$

Consider next the volume V_2, bounded as before except that we use the planes $x=h$ and $y=h$ instead of the cylinder $x^2+y^2=h^2$ (Fig. 2).

By double integration

$$V_2 = \int_0^h \int_0^h e^{-(x^2+y^2)}dxdy = \left[\int_0^h e^{-x^2}dx\right]^2.$$

Consider finally the volume V_3, bounded as for V_1 except that we use the cylinder $x^2+y^2=2h^2$ in lieu of $x^2+y^2=h^2$ (Fig. 3).

Again by the method of cylindrical shells $V_3=(\pi/4)[1-e^{-2h^2}]$.

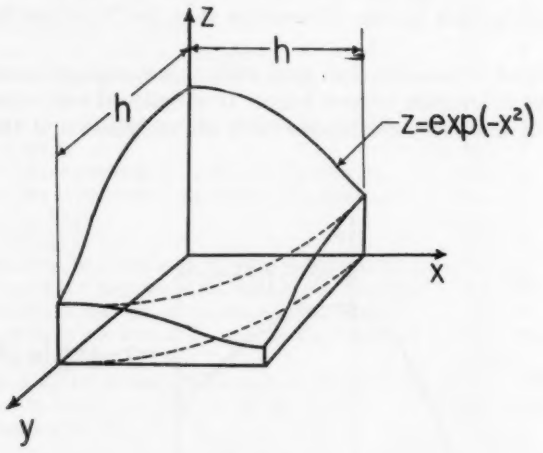

FIG. 2

Since by construction $V_1<V_2<V_3$, then for all positive values of h

$$\frac{\pi}{4}[1 - e^{-h^2}] < \left[\int_0^h e^{-x^2}dx\right]^2 < \frac{\pi}{4}[1 - e^{-2h^2}].$$

Now let h increase without bound, and seek limits of all three members. Evaluating the limits of the first and third members, we have

$$\frac{\pi}{4} \leqq \left[\lim_{h\to\infty} \int_0^h e^{-x^2}dx\right]^2 \leqq \frac{\pi}{4}.$$

Hence the middle limit must exist, and its value must be $\pi/4$. Therefore $\int_0^\infty e^{-x^2}dx=\frac{1}{2}\sqrt{\pi}$.

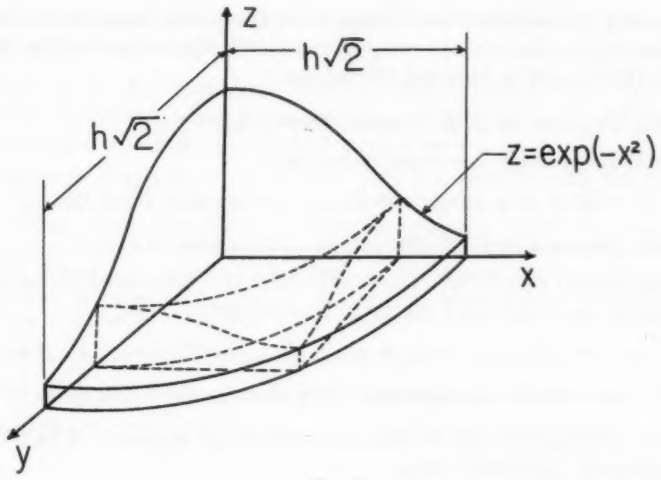

FIG. 3

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1291. *Proposed by G. B. Thomas, Jr., Massachusetts Institute of Technology*

Let a, b, c, d be nonnegative numbers such that $c+d \leqq \min(a, b)$. Prove that $ad+bc \leqq ab$.

E 1292. *Proposed by Jose Gallego-Diaz, Stanford University*

A pirate decided to bury a treasure on an island near the shore of which were two similar boulders A and B and, farther inland, three coconut trees C_1, C_2, C_3. Stationing himself at C_1, the pirate laid off C_1A_1 perpendicular and equal to C_1A and directed outwardly from the perimeter of triangle AC_1B. He similarly laid off C_1B_1 perpendicular and equal to C_1B and also directed outwardly from the perimeter of triangle AC_1B. He then located P_1, the intersection of AB_1 and BA_1. Stationing himself at C_2 and C_3, he similarly located points P_2 and P_3, and finally buried his treasure at the circumcenter of triangle $P_1P_2P_3$.

Returning to the island some years later, the pirate found that a big storm had obliterated all the coconut trees on the island. How might he find his buried treasure? (Dedicated to Howard Dachslager.)

E 1293. *Proposed by J. B. Roberts, Reed College*

Prove that for every nonnegative integer t

$$3(1 + 6^t + 8^t) \equiv 1^t + 2^t + 3^t + \cdots + 9^t \pmod{18}.$$

E 1294. *Proposed by G. A. Harris, Jr., Yale University*

Having chosen two numbers a_1 and b_1 from the open interval (0, 1), define the sequences $\{a_n\}$ and $\{b_n\}$ recursively as follows:

$$a_{n+1} = a_1(1 - a_n - b_n) + a_n, \qquad b_{n+1} = b_1(1 - a_n - b_n) + b_n.$$

Prove that both sequences approach limits as $n \to \infty$, and find these limits.

E 1295. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Show that all the roots of $\tan z = z/(1+m^2z^2)$, where m is real, are real.

SOLUTIONS

A Proposition Equivalent to Dirichlet's Theorem

E 1218 [1956, 342; 1957, 46]. *Proposed by Robert Spira, Berkeley, Calif.*

Consider the two propositions:

I. If $(a, b) = 1$, then $ax+b$ assumes infinitely many prime values.

II. If $(a, b) = 1$, then $ax+b$ assumes at least one prime value.

I is Dirichlet's theorem. Clearly I implies II. Show that II implies I.

II. *Solution by Virginia S. Hanly, Ohio State University.* Assume $|a| \neq 1$ and $(a, b) = 1$. The class $\{ax+b\}$ contains the two disjoint subclasses $\{a^2x+b\}$ and $\{a^2x+a+b\}$. Let p_1 be a prime of the form $ax+b$. Then p_1 is not in both of the subclasses (perhaps in neither). Hence, since $(a^2, b) = (a^2, a+b) = 1$, we let p_2 be a prime in one of the subclasses above not containing p_1. We may repeat the argument upon the subclass, obtaining an infinite sequence of distinct primes $p_1, p_2, \cdots$, all of the form $ax+b$. The case $|a| = 1$ follows trivially.

Editorial Note. Solutions previously received for this problem were patterned like that of [1957, 46], and become invalid if we are not restricted to positive integers. Thus, granted there is by II some x_1 such that ax_1+b is prime, if b is itself prime we cannot conclude that ax_1+b is distinct from b. In fact (see the solution [1957, 46]), letting $x_1 = x_2 = x_3 = \cdots = 0$, we have the sequence $b, b, b, \cdots$ containing just one prime. The new solution above was designed to avoid the restriction.

A Curve Defined by an Area Relation

E 1261 [1957, 272]. *Proposed by C. S. Ogilvy, Hamilton College*

If the sectorial area bounded by a curve and the radii vectores to any two

points on the curve equals the area bounded by the curve, the x-axis, and the ordinates of the two points, find the curve.

Solution by L. A. Ringenberg, Eastern Illinois State College. Equating the differentials of the two areas we have $|(xdy-ydx)/2| = |ydx|$, whence the required locus is any one of the curves $y=cx^2$ or $xy=c$.

Also solved by R. J. Adler, W. A. Al-Salam, J. W. Armstrong, Leon Bankoff, Robert Bart, D. A. Breault, Lawrence Conlon and Victor Manjarrez (jointly), L. R. Ford, David Freedman, Michael Goldberg, G. A. Harris, Jr., A. S. Hendler, Vern Hoggatt, J. R. Holdsworth, Bill Holter, R. H. Hou, A. R. Hyde, C. J. Kirchen, M. S. Klamkin, Frank Kocher, L. I. Lokomowitz, D. C. B. Marsh, E. J. Miller, John Osborn, G. B. Parrish, D. S. Passman, Montfort Plebstnoch, T. A. Porsching, John Rainwater, C. E. Rieck, D. A. Robinson, Anina Schub, Lawrence Shepp, J. A. Tierney, Roscoe White, T. L. Williams, R. H. Wilson, Jr., David Zeitlin, and the proposer.

Most of these solutions gave only one of the two classes of curves.

Cubics with Roots on the Unit Disc

E 1262 [1957, 272]. *Proposed by A. J. Goldman, National Bureau of Standards*

For which real values of A do all roots of $z^3-z^2+A=0$ obey $|z| \leqq 1$?

Solution by Arthur Steger, University of New Mexico. The graph of the real function $A=x^2-x^3$ has a relative maximum at $(2/3, 4/27)$, a relative minimum at $(0, 0)$, and passes through the points $(-1, 2)$, $(-1/3, 4/27)$, $(1, 0)$. From this graph we conclude the following concerning the roots of $z^3-z^2+A=0$:

(1) If $A<0$ or $A>2$, then there is a real root with absolute value greater than one.

(2) If $0 \leqq A \leqq 4/27$, then all the roots are real and lie between $-1/3$ and 1.

(3) If $4/27 < A \leqq 2$, then there is a real root between -1 and $-1/3$ and two imaginary roots. To investigate the imaginary roots, let r be the real root. Then $A=r^2-r^3$, and the depressed equation is $z^2+(r-1)z+r^2-r=0$ with roots

$$z = (1 - r \pm \sqrt{1 + 2r - 3r^2})/2.$$

For $-1 \leqq r < -1/3$, the radicand is negative and $|z|^2=r^2-r$. The condition $|z| \leqq 1$ yields $r^2-r \leqq 1$. Therefore $(1-\sqrt{5})/2 \leqq r < -1/3$. Finally, if $r=(1-\sqrt{5})/2$, then $A=(\sqrt{5}-1)/2$.

From (1), (2), and (3) we conclude that all the roots of $z^3-z^2+A=0$ have absolute value less than or equal to one if and only if

$$0 \leqq A \leqq (\sqrt{5}-1)/2.$$

Also solved by Robert Bart, Julian Braun, C. N. Campopiano, Michael Goldberg, A. R. Hyde, D. C. B. Marsh, G. B. Parrish, D. S. Passman, Marlow Sholander, E. P. Starke, and the proposer.

Sholander established the general theorem: *When a, b, c are real, the roots of $z^3+az^2+bz+c=0$ satisfy $|z| \leqq 1$, if and only if $|a+c|-1 \leqq b \leqq \min(3, 1+ac-c^2)$.* For $a=-1$, $b=0$, $c=A$, this reduces to the answer above.

Another Sequence of Triangles

E 1263 [1957, 272]. *Proposed by W. B. Carver, Cornell University*

Let T_0 be any triangle none of whose angles is a multiple of 45°. The tangents to the circumcircle of T_0 at its vertices form a new triangle T_1; and repetition of this process gives an infinite sequence of triangles $\{T_n\}$. If the angles of T_0 in degrees are integers, show that for $n \geqq 2$, T_{n+12} is similar to T_n. Is there a similar theorem for the case when the angles are rational in degrees?

Solution by D. C. B. Marsh, Colorado School of Mines. It is easy to show that: (1) if T_{n-1} is acute-angled, then $A_n = 180° - 2A_{n-1}$, $B_n = 180° - 2B_{n-1}$, $C_n = 180° - 2C_{n-1}$, (2) if T_{n-1} is obtuse-angled, say at A_{n-1}, then $A_n = 2A_{n-1} - 180°$, $B_n = 2B_{n-1}$, $C_n = 2C_{n-1}$. The fact that T_0 contains no angle which is a multiple of 45° guarantees that the construction process will not fail.

For any n, m we have $A_{n+m} \equiv 2^m A_n \pmod{180}$, for some choice of sign, no matter what sequence of the two recursive relations is involved. In the special case where $m = 12$ and $n \geqq 2$, we have

$$A_{n+12} \equiv \pm 2^{12} A_n \equiv \pm 2^{14} 2^{-2} A_n \equiv \pm 2^2 2^{-2} A_n \equiv \pm A_n \pmod{180}.$$

Similarly, for $n \geqq 2$,

$$B_{n+12} \equiv \pm B_n \pmod{180}, \qquad C_{n+12} \equiv \pm C_n \pmod{180},$$

where the same choice of sign occurs as for angle A_{n+12}. Since the sum of the angles of T_{n+12} and T_n is each 180°, it follows that, for $n \geqq 2$, $A_{n+12} = \pm A_n$, $B_{n+12} = \pm B_n$, $C_{n+12} = \pm C_n$, and T_{n+12} is (directly or inversely) similar to T_n.

If T_0 has angles rational in degrees, there is a corresponding theorem (which may be established analogously). Let the angles of T_0 have l.c.d. of $d = 2^e p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$, with the p_i distinct odd primes and $e_i \geqq 1$; let c_i be the exponent to which 2 belongs, mod p_i, and define E as the l.c.m. of 12 and the set of $c_i p_i^{e_i - 1}$; then, if no angle of T_0 is of the form $45a/2^b$ (a, b integers), the same method of construction yields T_{n+E} similar to T_n for $n \geqq 2 + e$.

Also solved (partially) by D. S. Passman.

Editorial Note. This sequence of triangles is something like an inverse of the sequence of triangles of problem E 1233 [vol. 64, 274]. If triangle T_{n-1} is acute-angled, its circumcircle will be the incircle of T_n, and the operation from T_{n-1} to T_n is exactly the inverse of the operation of problem E 1233. But in the sequence $\{T_n\}$ we sooner or later obtain an oblique triangle T_{k-1}, whose circumcircle will be, not the incircle, but one of the excircles of triangle T_k.

The Steensholt Inequality for a Tetrahedron

E 1264 [1957, 272]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If an interior point P of a tetrahedron $ABCD$ is projected orthogonally into A', B', C', D' on the planes of the faces BCD, CDA, DAB, ABC, and if the areas of these faces are denoted by A, B, C, D, show that

$$A(PA) + B(PB) + C(PC) + D(PD) \geqq 3[A(PA') + B(PB') + C(PC') + D(PD')].$$

I. *Solution by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.* Represent the volume of the tetrahedron by V. Then $V=(1/3)\sum A(PA')$. Also, $V=(1/3)h_A A$ and $h_A \leqq PA+PA'$. Hence $(1/3)\sum A(PA')=(1/12)\sum h_A A \leqq (1/12)\sum (PA+PA')A$, or $\sum A(PA) \geqq 3\sum A(PA')$.

II. *Solution by N. D. Kazarinoff, University of Michigan.* The proposed inequality follows from a theorem of Pappus generalized to E_3. This generalized theorem states: Construct three triangular prisms which have for their bases three faces of a tetrahedron T, which have a lateral edge in common, and of which all or none lie entirely outside of T; construct a fourth prism on the remaining face whose lateral edges are translates of the common lateral edge of the first three prisms; then, the sum of the volumes of the first three prisms is equal to the volume of the fourth prism. Apply this theorem to $ABCD$, using PA as the common lateral edge of the prisms constructed upon the faces having common vertex A. Then

$$A(PA)\cos(PA, A'P) = B(PB') + C(PC') + D(PD').$$

Replacing $\cos(PA, A'P)$ by 1 and combining the resultant inequality with the three others obtained from it by cyclic permutation, we obtain the desired inequality. Equality clearly holds if and only if $ABCD$ is orthocentric and P coincides with the orthocenter of $ABCD$.

Also solved by J. W. Armstrong, D. C. B. Marsh, C. S. Ogilvy, D. S. Passman, Chih-yi Wang, and the proposer.

Editorial Note. This problem extends to the tetrahedron a property of the triangle given by Gunnar Steensholt, this MONTHLY [1956, 571]. If the tetrahedron is isosceles (that is, equifacial), the inequality reduces to

$$PA + PB + PC + PD \geqq 3(PA' + PB' + PC' + PD'),$$

which establishes the Erdös-Mordell inequality for the tetrahedron ($2\sqrt{2}$ in place of 3) for this special type of tetrahedron.

The Trivial Function

E 1265 [1957, 272]. *Proposed by F. L. Wolf, Carleton College*

Is there a nonnegative, nontrivial, continuous function $f(x)$ such that

$$\int_0^x f(t)dt \geqq f(x)$$

for all x on $0 \leqq x \leqq 1$?

Solution by L. R. Ford, Illinois Institute of Technology. No. Excluding the trivial $f(x) \equiv 0$, suppose a nonnegative continuous $f(x)$ is not identically zero in $0 \leqq x \leqq b < 1$. Its maximum value $M > 0$ in this subinterval is the value of $f(x)$

at some point a therein. Then

$$\int_0^a f(t)dt \leqq Ma < f(a),$$

contrary to the inequality of the question.

Also solved by D. S. Adorno, W. A. Al-Salam, Robert Bart, F. G. Brauer, D. A. Breault, D. G. Brennan, J. L. Brown, Jr., R. G. Buschman, E. M. Cabaña, C. N. Campopiano, T. S. Chihara, R. M. Conkling, Frederic Cunningham, Jr., David Ellis, M. A. Feldstein, David Freedman, D. S. Greenstein, G. A. Harris, Jr., A. S. Hendler, Aaron Herschfeld, Vern Hoggatt, W. H. Holter, J. Horváth, R. H. Hou, A. R. Hyde, Ronald Jacobowitz, Seymour Kass, N. D. Kazarinoff, M. S. Klamkin, Paul Knopp and Victor Manjarrez (jointly), P. Kolar, L. I. Lokomowitz, Marshall Luban, L. A. MacColl, D. C. B. Marsh, C. T. Molloy, Jr., J. B. Muskat, Dale Nelson, C. S. Ogilvy, W. R. Orton and Harold Shniad (jointly), Hiram Paley and D. A. Robinson (jointly), G. B. Parrish, D. S. Passman, Edward Pincus, John Rainwater, C. E. Rieck, G. S. Rinehart, L. A. Ringenberg, Michael Rosen, Lawrence Shepp, W. G. Spohn, Jr., R. W. Wagner, David Zeitlin, and the proposer.

The problem was located as a special case of Prob. 1, p. 37, in Coddington and Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, 1955). Orton and Shniad established the answer *no* by weakening the hypothesis of continuity to Lebesgue integrability. Ogilvy pointed out that if the lower end of the interval is open, then the answer to the problem is *yes*.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4763. *Proposed by Ky Fan, Oak Ridge National Laboratory*

For n positive numbers x_i, let $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ denote respectively the least and the greatest of the $n+1$ quantities

$$x_1, \frac{1}{x_1} + x_2, \cdots, \frac{1}{x_{n-1}} + x_n, \frac{1}{x_n}. \tag{1}$$

Prove that

$$\operatorname*{Max}_{x_i>0} f(x_1, \cdots, x_n) = \operatorname*{Min}_{x_i>0} g(x_1, \cdots, x_n) = 2\cos\frac{\pi}{n+2}. \tag{2}$$

4764. *Proposed by A. E. Currier, U. S. Naval Academy*

Prove

$$\lim_{n\to\infty} \sum_{j=0}^{n} \binom{2n-j}{n} (-4)^j \Big/ \binom{2n}{n} = \frac{1}{3}.$$

4765. *Proposed by J. L. Massera, Institute of Mathematics and Statistics, Montevideo, Uruguay*

Let $y=f(x)$ be a real function defined for $x \geqq 0$. If (1) f has a finite upper bound in any finite interval, and (2) there are two positive numbers h, k such that $x'-x'' \leqq h$ implies $f(x')-f(x'') \geqq k$, then there is an increasing function $g(x)$ having as many continuous derivatives as we please, such that $g(x-h) < f(x) < g(x)$ for all sufficiently large x.

4766. *Proposed by Louis Weisner, University of New Brunswick, Canada*

Prove that the polynomials $g_n(x)$ defined by the generating function

$$(1+y^2)^{-1/2} e^{x \arctan y} = \sum_{n=0}^{\infty} \frac{g_n(x)}{n!} y^n, \qquad |y| < 1, \qquad g_0(x) = 1,$$

are orthogonal over the interval $[-\infty, \infty]$ with weight function $e^{\pi x/2}/(1+e^{\pi x})$. Prove that they satisfy the recurrence relations

$$g_{n+1}(x) = x g_n(x) - n^2 g_{n-1}(x); \qquad n = 0, 1, \cdots; g_{-1}(x) = 0.$$

4767. *Proposed by J. L. Brenner, Stanford Research Institute*

Let a, b, d be integers, w real, $w \geqq 1+\sqrt{2}$, $d > wa > w^2 b > 0$, $a^2 - bd = 1$. Show that

$$A = \begin{pmatrix} a & b \\ d & a \end{pmatrix}, \qquad B = \begin{pmatrix} a & d \\ b & a \end{pmatrix}$$

generate a free group.

SOLUTIONS

Irreducible Polynomials

4709 [1956, 669]. *Proposed by R. W. Marsh, Arlington, Va., and A. M. Gleason, Harvard University*

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be an irreducible polynomial with coefficients in the finite field of q elements. Suppose that the roots of f are primitive; that is, they generate the multiplicative group of their extension field. Prove that

$$F(x) = x^{q^n-1} + a_{n-1}x^{q^{n-1}-1} + \cdots + a_0$$

is irreducible.

Solution by Leonard Carlitz, Duke University. This result is due to Ore, *Contributions to the Theory of Finite Fields*, Trans. Amer. Math. Soc., vol. 36, 1934, p. 260, Theorem 1. An outline of the proof follows. In view of the hypothesis on $f(x)$, q^n-1 is the smallest positive exponent such that

$$x^{q^n-1} - 1 \equiv 0 \pmod{f(x)};$$

this implies that the smallest positive exponent s such that

$$x^{q^s} - x \equiv 0 \pmod{F(x)}$$

is given by $s=q^n-1$. Consequently $F(x)$ is irreducible.

Also solved by the proposers.

Pascal Hexagon and Cubic

4718 [1957, 49]. *Proposed by V. F. Ivanoff, San Carlos, California*

The six points of intersection of a conic and a cubic determine a Pascal hexagon. Show that the residual six points of intersection of the sides of the hexagon with the cubic form two collinear sets, and the lines determined by these sets meet on the Pascal line.

Solution by O. J. Ramler, Catholic University of America. Let the conic Γ cut the cubic Σ in the six points A, B, C, D, E, F which we take as a simple (Pascal) hexagon. Let AB meet DE in L, BC meet EF in M, CD meet FA in N. Then LMN is the Pascal line for this hexagon. Let the sides AB, CD, EF BC, DE, FA meet the cubic again in X, Y, Z, X', Y', Z', respectively. For brevity, let $\Sigma=0$, $\Gamma=0$, $(AB)=0$ represent the equations of the cubic, the conic, and the line AB.

We observe that $\Sigma+k(AB)(CD)(EF)=0$ is the equation of a system of cubics through the nine points $A, B, C, D, E, F, X, Y, Z$. Since six of these points are on the conic Γ, it follows that the remaining three, X, Y, Z, are collinear. A similar procedure shows X', Y', Z' collinear. Now consider the equation

$$(AB)(CD)(EF)(X'Y'Z') + k(BC)(DE)(FA)(XYZ) = 0,$$

which represents a pencil of quartics through the sixteen points $A, B, C, D, E, F, X, Y, Z, X', Y', Z', L, M, N, P$, where P is the intersection of lines XYZ and $X'Y'Z'$. However, the first twelve of these points are on the cubic Σ, hence the remaining four, L, M, N, P, are collinear, *i.e.* the lines XYZ, $X'Y'Z'$, LMN are concurrent as required.

Also solved by J. Basile, R. Deaux, R. C. Lyness, and E. J. F. Primrose.

A Determinant of Legendre Symbols

4719 [1957, 49]. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime >2. Show that the determinant of order $p-1$

$$\Delta_p = \left|\left(\frac{r-s}{p}\right)\right| \qquad (r, s = 0, 1, \cdots, p-2),$$

where (r/p) is the Legendre symbol, satisfies $\Delta_p = p^{(p-3)/2}$.

Solution by the proposer. The well known formula for a circulant

$$\left| x_{r-s} \right|_{r,s=0,\cdots,p-1} = \prod_{s=0}^{p-1} \sum_{r=0}^{p-1} \epsilon^{rs} x_r \quad (\epsilon = e^{2\pi i/p})$$

yields, when $\sum x_r = 0$,

$$\left| x_{r-s} \right|_{r,s=0,\cdots,p-2} = \frac{1}{p} \prod_{s=1}^{p-1} \sum_{r=0}^{p-1} \epsilon^{rs} x_r. \tag{1}$$

Now take $x_r = (r/p)$; then (1) becomes

$$\Delta_p = \frac{1}{p} \prod_{s=1}^{p-1} \sum_{r=1}^{p-1} \left(\frac{r}{p}\right) \epsilon^{rs}. \tag{2}$$

Next we recall that the Gauss sum $G(s)$ satisfies

$$G(s) = \prod_{r=1}^{p-1} \left(\frac{r}{p}\right) \epsilon^{rs}, \qquad G(s) = \left(\frac{s}{p}\right) G(1), \qquad G^2(s) = \left(\frac{-1}{p}\right) p.$$

Thus (2) gives

$$\begin{aligned} \Delta_p &= \frac{1}{p} \prod_{s=1}^{p-1} G(s) = \frac{1}{p} \prod_{s=1}^{p-1} \left\{\left(\frac{s}{p}\right) G(1)\right\} \\ &= \frac{1}{p} (-1)^{(p-1)/2} \left\{\left(\frac{-1}{p}\right) p\right\}^{(p-1)/2} \\ &= \frac{1}{p} \left(\frac{-1}{p}\right) \cdot \left(\frac{-1}{p}\right) p^{(p-1)/2} = p^{(p-3)/2}. \end{aligned}$$

Also solved by N. J. Fine.

Coefficients in a Power Series

4720 [1957, 49]. *Proposed by S. W. Golomb, University of Oslo, Norway*

Show that in the power series expansion

$$\frac{1}{1 - 2x - 2x^2 + x^3} = \sum_{n=0}^{\infty} a_n x^n, \qquad |x| < 1,$$

the coefficients are given by $a_n = \sum_{k=0}^{n} f_k^2$, where $f_0 = 1, f_1 = 1, f_2 = 2, \cdots$ is the Fibonacci sequence.

Solution by Peter Henrici, University of California, Los Angeles. The desired conclusion is equivalent to

$$\frac{1-x}{1-2x-2x^2+x^3} = \sum_{n=0}^{\infty} f_n^2 x^n$$

or, multiplying by the denominator and comparing coefficients, ($f_{-1}=f_{-2}=\cdots=0$),

$$f_n^2 - 2f_{n-1}^2 - 2f_{n-2}^2 + f_{n-3}^2 = \begin{cases} 1, & n = 0, \\ -1, & n = 1, \\ 0, & n \geqq 2. \end{cases} \tag{1}$$

This relation is evident for $n=0, 1$; to prove it for $n \geqq 2$, we have

$$\begin{aligned} f_n^2 &= (f_{n-1}+f_{n-2})^2 = 2f_{n-1}^2 + 2f_{n-2}^2 - (f_{n-1}-f_{n-2})^2 \\ &= 2f_{n-1}^2 + 2f_{n-2}^2 - f_{n-3}^2, \end{aligned}$$

as desired.

Note that the roots of $1-2x-2x^2+x^3=0$ are $1, \frac{1}{2}(3\pm\sqrt{5})$, so that the radius of convergence is $\frac{1}{2}(3-\sqrt{5})<1$.

Also solved by J. L. Alperin, Anders Bager, J. D. Baum and Samuel Goldberg, Louis Brand, Robert Breusch, W. E. Briggs, P. L. Chessin, Charles Conley, M. Delcourt, J. S. Denton, Jr., P. L. Duren, N. J. Fine, M. F. Friedell, H. E. Goheen, Cornelius Groenewoud, Emil Grosswald, Joseph Hershenov, L. N. Howard, Richard Kelisky, M. S. Klamkin, R. C. Lyness, D. C. B. Marsh, Yoshio Matsuoka, Norman Miller, J. B. Muskat, Živadin Pantić, F. D. Parker, B. I. Penkov, R. S. Pinkham, R. C. Read, L. A. Ringenberg, D. A. Robinson, Lawrence Shepp, P. Somanadham, J. R. Trollope, Chih-yi Wang, D. G. Wertheim, L. K. Williams, David Zeitlin, and the proposer.

A Function Summable over the Entire Plane

4721 [1957, 49]. *Proposed by D. J. Newman, AVCO Research and Advanced Development, Lawrence, Mass.*

Let $u(x, y)$ be continuous and summable ($\iint |u|\,dxdy < \infty$) over the entire plane. Suppose the line integral, $\int_L u(x, y)ds$, vanishes for all straight lines L infinite in both directions. Prove that $u(x, y)$ is identically zero.

Solution by the proposer. We prove $u(0, 0)=0$. By the symmetry of the problem this will suffice. First of all, we have

$$S_m = \int u(x, y)ds = 0,$$

where the integration is over the line $y \cos\theta = x \sin\theta + m$, so that

$$\int_{-\infty}^{\infty} e^{-Nm^2} S_m dm = 0.$$

Now we have, writing this as a double integral,

$$\iint_{\text{plane}} e^{-N(y\cos\theta - x\sin\theta)^2} u(x, y)dxdy = 0.$$

Next, integrate with respect to θ from 0 to 2π. Upon interchanging the order we have

$$\iint_{\text{plane}} u(x, y)F_N dxdy = 0, \qquad F_N = N\int_0^{2\pi} e^{-N(y\cos\theta - x\sin\theta)^2} d\theta.$$

But the integral F_N is equal to $2\pi N e^{-Nr^2/2} I_0(Nr^2)$, where I_0 is the Bessel function and $r = (x^2+y^2)^{1/2}$.

Now we employ the δ-function-like behavior of F_N, namely

$$\iint_{\text{plane}} F_N = C \neq 0; \qquad \iint_{r>\delta} F_N \to 0, \qquad N \to \infty.$$

It follows immediately that

$$\iint_{\text{plane}} u(x, y)F_N dxdy \to C \cdot u(0, 0).$$

Since this integral is identically zero, it follows that $u(0, 0) = 0$.

Editorial Note. As remarked by B. I. Penkov, the theorem of the problem is a generalization of a theorem of Radon and is stated and solved in A. Renyi, *On projections of probability distributions*, *Acta Math. Acad. Sci. Hungar.*, vol. 3, 1952, fasc. 3, pp. 131–141. See also, *Mathematical Reviews*, vol. 15, 1954, p. 139.

Hölder's Inequality

4722 [1957, 49]. *Proposed by R. C. Warner, Toronto, Canada*

Establish the following inequality

$$\left\{\sum_{i=1}^{n}\prod_{j=1}^{r} a_{ij}^{m/r}\right\}^r \leqq \prod_{j=1}^{r}\sum_{i=1}^{n} a_{ij}^{m}.$$

Editorial Note. As pointed out by Chih-yi Wang and O. Mourmaki, this is the special case of Hölder's inequality in which every exponent has been made equal to $1/r$. There is no loss in generality in taking $m = 1$. For proofs and a number of generalizations see Hardy, Littlewood and Pólya, *Inequalities*, Cambridge, 1934, pp. 21–26.

Also solved by Robert Breusch, N. J. Fine, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Studies in Differential Equations. Ed. by Harold T. Davis, Northwestern University, 1956. 114 pp. $1.75.

Studies in Differential Equations is a collection of papers by Harold T. Davis, George Springer, W. T. Scott, and Daniel Resch. The first half of the study is concerned with certain special ordinary differential equations of the second order. The second half of the study deals with the general topic of Baecklund Transformations.

In the first part, the nonlinear equation

$$A(y)\frac{d^2y}{dx^2} + B(y)\frac{dy}{dx} + C(y)\left(\frac{dy}{dx}\right)^2 + D(y) = 0 \tag{1}$$

is considered. Special cases of this equation are the Van der Pol Equation and the Volterra Equation. After a general survey of the field we are introduced to the six Painlevé Equations which are special cases of equation (1). The simplest of the Painlevé Equations is the equation

$$\frac{d^2y}{dx^2} = \sigma y^2 + \lambda x. \tag{2}$$

Solutions of the Painlevé Equations are called Painlevé Transcendents. Much of this section of the book is devoted to the study of Painlevé Transcendents. Numerical solutions are obtained for a number of different values of the parameters involved. While the topic of Painlevé Transcendents deals with certain special cases of equation (1), the section dealing with linear fractional transformations of the dependent variable in (1) is of general validity. A number of theorems dealing with invariant properties of (1) under linear fractional transformations are interesting.

The second part of the volume deals with the use of Baecklund Transformations in partial differential equations. A Baecklund Transformation is defined by the four equations

$$F_i(x, y, z, z_x, z_y, x', y', z', z'_{x'}, z_{1.}) = 0,$$

where z and z' are unknown functions of (x, y) and (x', y') respectively. If this system can be solved, it defines not only a surface Σ in (x, y, z)-space and a surface Σ' in (x', y', z')-space, but also a correspondence (but not usually a one-one mapping) between the two. This transformation is called a Baecklund Transformation. The notion of a Baecklund Transformation can be extended to higher dimensions. In this volume, George Springer investigates the Baecklund

Transformations of a certain type under which a system of first order partial differential equations remains invariant. W. T. Scott establishes the existence, under suitable conditions, of linear Baecklund Transformations linking systems of second order equations. Daniel Resch investigates Baecklund Transformations which will link second order partial differential equations in three independent variables. It is shown in particular that if the Baecklund Transformation is linear and in $z, z', z_x, z_y, x'_{x'}, z'_{y'}$, then second order equations can be linked to the wave equations.

As can be seen from the description, the authors deal with a variety of topics of interest. The "*Study*" should be of use to individuals who work in either the pure or the applied branches of differential equations.

F. Haas
Wayne State University

Mathematical Logic. By R. L. Goodstein. Leicester University, 1957. viii+104 pp. 21/-.

"The aim of this little book is to introduce teachers of mathematics to some of the remarkable results which have been obtained in mathematical logic during the past twenty-five years." Probably the most interesting of these results are the Gödel completeness and incompleteness theorems and some of their consequences. The book is essentially self-contained, and the first two chapters outline the necessary background in the sentence and predicate calculus ending with a detailed proof (Henkin's) of the Gödel completeness theorem for the predicate calculus. Many other topics, such as Intuitionistic logic, bracket-free notation, are briefly treated. In every case the author illustrates the ideas involved by one or more examples. Chapter III introduces formal number theory (Z) and discusses various facets of recursive function theory. At the end of this chapter, he begins the proof of the Gödel incompleteness theorem by proving that recursive functions are, essentially, expressible in Z. In the next chapter, Z plus its syntax is mapped into intuitive number theory via a Gödel numbering and, in quite reasonable detail, it is shown that appropriate parts of the syntax are taken into recursive functions. Assuming ω-consistency, Z is then proved incomplete. The concept of ω-consistency is further illustrated by constructing a nonstandard model for Z. The chapter closes with a proof that Z and predicate calculus do not have a decision procedure. In the last chapter a class logic—specifically Quine's *Mathematical Logic*—is described, and the sequence of definitions leading to a formalization of number theory in this system is given.

It is clear that the author has covered a great deal of interest in 100 pages. Of particular interest is the treatment of the incompleteness theorem, since "semi-popular" accounts often leave much to be desired in, for example, discussing the Gödel numbering. The mere fact that Z plus its syntax is countable is trivial; the important thing is the connection between recursive definitions and recursive functions made possible by a Gödel numbering. The author includes sufficient detail so that careful reading (and careful reading will probably

be required of its intended class of readers) puts such things in evidence. However, on another level the book leaves some things to be desired. The book does not give much meaning to the particular undecidable sentence used (and it is probably too much to expect of the reader to supply this), or, more important, an overall guide to the Gödel theorem such as given in the introduction to Mostowski's *Sentences Undecidable in Formalized Arithmetic*. Similarly, the non-standard model for Z may be disquieting in view of the general feeling that the Peano Axioms are categorical,—it would have been easy to point out that this depends on the relatively narrow class of sentences allowed for induction in Z. However, such remarks can not alter the fact that the author has written a commendable and worthwhile book.

Misprints appear to be very scarce and most of them are minor. However, the reviewer is convinced that the definition of $Sb(e, t, x)$ on page 76 is incorrect, —for example, instead of $e=2^{13}\cdot 3^{n}\rightarrow Sb(e,\ t,\ x)=2^{13}\cdot 3^{Sb(n,t,x)}$ it should be $e=2^{13}\wedge B(n)\rightarrow Sb(e, t, x)=2^{13}\wedge B(Sb(n, t, x))$. Analogous remarks apply to the definition of $t(n)$. (Reason: the Gödel numbering used assigns odd integers to signs, even integers whose prime power factorization uses odd exponents to sequences of signs or sentences, and even integers with even powers for the primes to sequences of sentences.)

J. B. GIEVER
The University of Oklahoma

Advanced Calculus. By R. Creighton Buck. McGraw-Hill, New York, 1956. vi+423 pages. $8.50.

The work under review may be described briefly, and perhaps with fair accuracy, by saying that it is rigorous, carefully composed, easy reading for the most part, more unified than the majority of texts on the subject, modernistic, and advanced in viewpoint. These last two characteristics are major marks of distinction and give the book a definite individualistic character.

A fair proportion of the material and methods presented are drawn from fields that are either relatively new or are currently in vogue for study and research. For example, a variety of topological concepts are introduced, and there is heavy emphasis on mappings and transformations of various kinds. In addition, there is included an exposition of the algebra and calculus of Cartan's exterior differential forms together with applications. Thus the book has more twentieth century overtones than most treatments of advanced calculus, and it is in this sense that it may be said to be modernistic. The material on the various meanings for the symbols *dx etc.* in differential and integral calculus, together with the discussions on change of variables in integrals, should be a good antidote for the semantic difficulties inherent in the archaic use of the notation in some scientific writings.

This text is advanced with regard to viewpoints, in the sense that the language and formulations used in certain fundamental definitions, while not abstruse and recondite, emphasize aspects which are merely implicit in the usual

statements. The superiority of some of these departures from the traditional is immediately apparent, as witness:

"**Definition.** *An infinite series of real numbers is a pair of real sequences* $\{a_n\}$ *and* $\{A_n\}$ *whose terms are connected by the relations*:

$$A_n = \sum_1^n a_k = a_1 + a_2 + \cdots + a_n \tag{4-1}$$

$$a_1 = A_1, \qquad a_n = A_n - A_{n-1} \qquad n \geqq 2." \text{ (p. 105).}$$

In other cases class-room discussions may be necessary for clarification or to bring out the advantages of the formulation given, as for example:

"**Definition.** *A curve* γ *in n-space is a mapping or transformation from* E^1 *to* E^n." (p. 251.)

"**Definition.** *Let f be any function of class C' in a region D of n-space. Then, the differential of f at a point* $p \in D$ *is the linear function L of n variables specified by the matrix* $[f_1(p), f_2(p), \cdots, f_n(p)]$.

When we write $w=f(x, y)$, for example, the differential of f is given by $[\partial w/\partial x, \partial w/\partial y]$, and its value at a point $(\Delta x, \Delta y)$ is $(\partial w/\partial x)\Delta x+(\partial w/\partial y)\Delta y$." (p. 184.)

In one particular instance, it seems to the reviewer that a change in the order of the material presented would be desirable. On pages one and two the dot product is introduced as the inner product of two points $p(x_1, \cdots, x_n)$, $q(y_1, \cdots, y_n)$ in the form

$$p \cdot q = x_1y_1 + \cdots + x_ny_n,$$

without mention of the nature of the coordinate system or the question of invariance, and on page 328 it is stated "the inner product of two points, p and q is an algebraic function of their coordinates whose value is unchanged if the underlying 3-space is subjected to any orthogonal linear transformation (rotation) resulting in new coordinates for p and q."

To sum up, this is not just an ordinary textbook presenting the usual material in the usual way, but one that is fresh and modern in many respects. It should prove to be a very useful and welcome addition to the literature on its subject.

HOMER V. CRAIG
The University of Texas

BRIEF MENTION

The Logical Problem of Induction (2nd Edition). By G. H. Von Wright. Macmillan, New York, 1957. xii+249 pp. $4.00.

This revision of Von Wright's earlier (1941) dissertation on Hume's problem will be of interest to students of the traditional problem of the justification of induction.

Elementary Theory of Angular Momentum. By M. E. Rose. Wiley, New York, 1957. x+248 pp. $10.

This clear presentation of the modern theory of angular momentum based on quantum mechanics is the outgrowth of a series of lectures given at Oak Ridge National Laboratory.

Experimental Designs (2nd Edition). By William G. Cochran and Gertrude M. Cox. Wiley, New York, 1957. xiv+617 pp. $10.25.

It is always encouraging to find a second edition of a nonelementary text. This presentation of statistical experimental design should be welcomed by scientists in many fields.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

FELLOWSHIP AND RESEARCH OPPORTUNITIES

The Division of Mathematics of the National Academy of Sciences—National Research Council wishes to call attention to the fact that several foundations and offices will offer financial support for research in mathematics during the year 1958–59. A number of fellowships will be made available, as well as opportunities for mathematicians to engage in basic research. A partial list, with comments, is given below.

1. *National Science Foundation. Fellowships.*

Predoctoral fellowships are awarded annually at the First Year, Intermediate, and Terminal Year levels of graduate study. Applications for 1958–1959 are available from the National Academy of Sciences—National Research Council until the closing date in early January 1958; award date—March 15, 1958.

Science Faculty fellowships for college science teachers (including mathematics) who plan to continue teaching and wish to increase their competence as teachers are at the present time offered semi-annually. Eligibility requirements include a baccalaureate degree and three years of full-time experience in teaching natural science subjects at the collegiate level. The program was opened to application in October 1957 and will be closed early in January 1958. Awards will be made on March 20, 1958. The program will also be reopened the following summer. Address all inquiries for information and applications to National Science Foundation, Division of Scientific Personnel and Education, Washington 25, D. C.

Regular postdoctoral fellowships, primarily for recent recipients of the doctoral degree, are awarded semi-annually. Program for 1958–1959 is concurrent with above predoctoral program except that program closes in December. Information and applications are available from NAS-NRC. The program will also be open from July to early September 1958. Awards are announced in March and October.

Senior postdoctoral fellowships, which are open to persons who have held a doctoral degree in one of the basic fields of science for a minimum of five years at time of application, or who have had equivalent training and experience, are awarded semi-annually. Applications are available from the National Science Foundation, Division of Scientific Personnel and Education, Washington 25, D. C. The program is open from October 1957 until January 1958. Awards will be announced on March 18, 1958. The program will be reopened in the summer of 1958.

Research Grants. The National Science Foundation also supports basic research in the mathematical sciences by means of grants. While proposals for such support are accepted at any time, individuals desiring support to begin in the summer or at the beginning of a fall semester should preferably submit their proposals in the mathematical sciences by November 1; persons desiring support to begin in the spring semester should preferably submit their proposals by May 1. Instructions for the preparation of proposals, contained in a booklet entitled Grants for Scientific Research, may be obtained upon request from the Program Director for Mathematical Sciences, National Science Foundation, 1520 H Street, N. W., Washington 25, D. C.

2. *Office of Naval Research.* The Office of Naval Research, through contracts with universities and other organizations, supports basic research in broadly selected fields of mathematics. Proposals should be directed to the Mathematics Branch, Office of Naval Research, Washington 25, D. C. In addition, postdoctoral research associateships in pure mathematics are being established under contracts with the ONR at selected universities. For details and application forms write to the above address.

3. *Air Force Office of Scientific Research.* The Air Force Office of Scientific Research supports research in mathematics directly through contracts with colleges, universities, foundations and industrial laboratories. Such organizations are encouraged to submit proposals for research in mathematical fields in which they specialize. Proposals should be mailed to the Commander, Air Force Office of Scientific Research, Attn: Mathematics Division, Washington 25, D. C.

4. *Office of Ordnance Research, U. S. Army.* Among the functions of the Office of Ordnance Research is the support of basic research in mathematics. Proposals for projects are ordinarily made by individual scientists or groups of scientists in a form which leads to a contract between the Office of Ordnance Research and a university or research laboratory. For further information write to Commanding Officer, Office of Ordnance Research, Box CM, Duke Station, Durham, North Carolina.

5. *Fulbright Awards—Public Law 584 (79th Congress).* Approximately 400 awards are offered annually for university lecturing and postdoctoral research in all academic fields in Australia, Burma, Chile, Colombia, Ecuador, India, New Zealand, Pakistan, Paraguay, Peru, the Philippines, Thailand (competition for the preceding countries closes April 15, 1958); Austria, Belgium-Luxembourg, Denmark, Finland, France, Germany, Greece, Iceland, Ireland, Israel, Italy, Japan, the Netherlands, Norway, Turkey, and the United Kingdom including colonial dependencies (competition for the latter countries closes October 1, 1958). In both cases awards are for the academic year 1959–60, but in the former group of countries the academic year begins in the spring or summer instead of the autumn. Awards are payable in foreign currency and usually include travel for the grantee, but not for members of his family, and a maintenance allowance, which may be adjusted in relation to the number of accompanying dependents up to four. Requests for information should be addressed to the Committee on International Exchange of Persons, Conference Board of Associated Research Councils, 2101 Constitution Avenue, Washington 25, D. C.

6. *National Bureau of Standards; Naval Research Laboratory; Oak Ridge National Laboratory.* Postdoctoral resident research associateships are available in a variety of sciences including mathematics, and are tenable at the Washington, D. C. and Boulder, Colorado laboratories of the National Bureau of Standards; at the Naval Research Laboratory in Washington, D. C.; and at the Oak Ridge National Laboratory in Oak Ridge, Tennessee. Necessary facilities and equipment incident to the research of the Associate will be provided. For further information write to Fellowship Office, National Academy of Sciences-National Research Council, 2101 Constitution Avenue, Washington 25, D. C. Applications for the 1958–59 program must be filed on or before January 13, 1958.

7. *Atomic Energy Commission.* The Division of Research of the Atomic Energy Commission through contracts with universities and other organizations supports research in the fields of numerical analysis, digital computer design, programming research, and related topics. Proposals should be submitted to the Division of Research, Atomic Energy Commission, 1901 Constitution Avenue, Washington 25, D. C.

Brookhaven National Laboratory. Brookhaven National Laboratory, operated by Associated Universities, Inc. under contract with the Atomic Energy Commission offers postdoctoral research appointments in mathematics. Appointments are for one year, and may be renewed for one additional year. U. S. citizenship is not required, although Atomic Energy Commission approval is a prerequisite. The appointee may work in numerical analysis, digital computing, mathematical physics, differential equations, probability and statistics, and various specialized branches including reactor theory, hydrodynamics, and orbit theory. Computational facilities are available. Letters from candidates should give details of personal history, scientific background, and qualifications; two letters of recommendation, one from the applicant's research professor, are required. Applications for the academic year 1958–59 must be received by August 15, 1958 and should be directed to M. E. Rose, Head, Applied Mathematics Division, Brookhaven National Laboratory, Upton, Long Island, New York.

PERSONAL ITEMS

Professor G. B. Price, University of Kansas, represented the Association at the Fortieth Annual Meeting of the American Council on Education in Washington, D. C., on October 10–11, 1957.

Boston College: A Workshop for High School Teachers of Mathematics was conducted during the summer; the speakers included Dean A. E. Meder, Jr., J. B. Adkins, John Kemeny, Max Beberman, and Rev. S. J. Bezuszka.

Butler University: Capt. F. A. Graf (USN, Ret.), formerly Director of the Naval Observatory, and Miss Joyce Cimelus, Statistics Laboratory, Purdue University, have been appointed Instructors.

Florida State University: Dr. M. B. Smith, Jr., University of North Carolina, has been appointed Assistant Professor; Mr. J. F. Brooks and Mr. G. W. Polites, Graduate Assistants at the University, have been appointed Instructors; Assistant Professors J. W. Ellis and H. C. Griffith have been promoted to Associate Professors.

Fordham University: Mr. N. C. Mitrowsis, Graduate Assistant, University of South Carolina, has been appointed Instructor; Assistant Professor W. T. Shields has retired.

Georgetown University: Mr. Henry Beiman, University of Maryland, has been appointed Instructor; Mr. J. E. Houle has been promoted to Assistant Professor; Rev. F. W. Sohon has retired with the title Professor Emeritus.

Hampton Institute: Assistant Professor Rosalind Eagleson has been promoted to Associate Professor; Miss Harriett R. Junior, Assistant, Syracuse University, has returned to the Institute.

Indiana University: Assistant Professor Louis Auslander, University of Pennsylvania, and Dr. Werner Gautschi, Ohio State University, have been appointed Assistant Professors; Dr. Avner Friedman, Research Associate, University of Kansas, has been appointed Lecturer; Professor Ratip Berker, Technical Institute of Istanbul, has been appointed Visiting Professor for one semester.

Iowa State College: Dr. M. F. Ruchte, Yale University, has been appointed Assistant Professor; Mrs. William Proett, Assistant, University of Texas, has been appointed Instructor.

Kansas State College: Dr. B. G. Pearson, Illinois Institute of Technology, and Assistant Professor Evelyn Kinney, Illinois Wesleyan University, have been appointed Assistant Professors; Miss Lorraine Schwartz, Graduate Student, University of California, has been appointed Instructor; Assistant Professors A. M. Feyerherm, and W. L. Stamey have been promoted to Associate Professors.

Kent State University: Mr. Kenneth L. Cummins, Teacher, New Washington High School, Ohio, has been appointed Assistant Professor; Miss Grace L. Abhau and Mr. R. F. Liskovec have been appointed Instructors.

McMaster University: Dr. J. H. H. Chalk, Lecturer, Bedford College, University of London, has been appointed Assistant Professor; Assistant Professors F. R. Britton, Bernard Banaschewski, and N. D. Lane have been promoted to Associate Professors.

New Mexico College of Agriculture and Mechanic Arts: Assistant Professor E. A. Walker, University of Kansas, and Dr. K. R. Lucas, Graduate Student, University of Kansas, have been appointed Assistant Professors; Assistant Professor R. M. Conkling has been promoted to Associate Professor.

Princeton University: Associate Professor E. A. Coddington, University of California, Los Angeles, has been appointed Visiting Associate Professor; Dr. J. P. Roth, International Business Machines Corporation, Poughkeepsie, New York, has been appointed Visiting Assistant Professor; Dr. G. O. Losey, Teaching Fellow, University of Michigan, Dr. R. D. McWilliams, University of Tennessee, and Dr. James Munkres, University of Michigan, have been appointed Instructors; Dr. Ralph Gomory, Office of Naval Research, Washington, D. C., has been appointed Higgins Lecturer; Assistant Professor I. P. Guttman, who is on leave of absence from the University of Alberta, has been appointed Research Associate; Associate Professor Valentin Barmann has been promoted to Professor; Assistant Professor John Milnor and John Moore have been promoted to Associate Professors.

State University of Iowa: Associate Professor Byron Cosby, Jr., who was on leave of absence during 1956–57, has returned; Professor Emeritus E. W. Chittenden has been granted a leave of absence for 1957–58 in order to continue his appointment as mathematician in the Diamond Fuse Laboratory, Washington, D. C.; Professor Emeritus C. C. Wylie has been granted a leave of absence for the academic year 1957–58 to accept an appointment at Park College.

Stevens Institute of Technology: Dr. Lawrence Goldman, Columbia University, has been appointed Assistant Professor; Mr. Henry Polowy and Mr. M. E. White have been promoted to Assistant Professors.

Technical Operations, Monterey, California: Mr. W. S. Cox, Mr. W. J. Harrington, and Miss Eleanor C. Shlifer have been appointed Operations Analysts.

United States Naval Academy: Mr. R. E. Walters, University of Louisville, has been appointed Assistant Professor; Associate Professors E. E. Betz and S. S. Saslaw have been promoted to Professors; Assistant Professors M. V. Gibbons, E. C. Gras, and W. J. Strange have been promoted to Associate Professors.

University of Alabama: Assistant Professors J. H. Hornback, B. M. Seelbinder, H. C. Filgo, and C. C. Buck have been promoted to Associate Professors.

University of Alberta: Dr. A. P. Guinand, University of New England, Australia,

Dr. S. S. Gupta, Bell Telephone Laboratories, Pennsylvania, and University of North Carolina, and Dr. H. F. J. Lowig, University of Tasmania, have been appointed Associate Professors; Dr. I. N. Baker, Assistant Professor G. C. Cree, University of Nebraska, and Dr. E. L. Whitney, Operational Research Group, Ottawa, have been appointed Assistant Professors; Miss Z. M. Hyduk, Graduate Student at the University, has been appointed Sessional Lecturer; Assistant Professor H. Helfenstein, who is on leave of absence for 1957–58, is a staff member of the University of Ottawa.

University of Arizona: Assistant Professor Fay Farnum, Iowa State College, has been appointed Assistant Professor; Mr. H. L. Hancock, E. B. Hoff, G. M. Jones, J. D. Simley, and A. H. Tellez have been appointed Instructors; Mr. J. E. Lee, Mr. J. E. Strang, and Mrs. E. Virginia Prebula have been appointed Teaching Assistants; Assistant Professor Deonisie Trifan has been promoted to Associate Professor.

University of Arkansas: Dr. J. E. Scroggs has been appointed Assistant Professor; Miss Marion Brashears has been appointed Instructor; Assistant Professor W. R. Orton has been promoted to Associate Professor.

University of British Columbia: Dr. D. W. Bressler, Graduate Student, University of California, and Dr. C. A. Swanson, Teaching Assistant, California Institute of Technology, have been appointed Instructors; Miss Charlotte Froese, and Assistant Professor R. R. Struik, Drexel Institute of Technology, have been appointed Lecturers; Dr. R. R. Christian and Dr. H. F. Davis have been promoted to Assistant Professors.

University of Colorado: Dr. Irwin Fischer, Dartmouth College, has been appointed Assistant Professor; Mrs. Dina G. Thomas, Part-time Lecturer, Boston University, and Part-time Instructor, Suffolk University, has been appointed Instructor; Associate Professor W. J. Thron has been promoted to Professor; Dr. Aboulghassem Zirakzadeh has been promoted to Assistant Professor; Professor Claribel Kendall has retired with the title Professor Emeritus; Professor W. W. Rogosinski, King's College, University of Durham, England, is replacing Professor Thron, who is on leave of absence at the University of Munich, Germany, under an Air Force contract; Professor Sigmund Selberg, Institute of Technology, Trondheim, Norway, is replacing Professor Sarvadaman Chowla, who is on leave of absence at the Institute for Advanced Study.

University of Connecticut: Mrs. Helen G. Brown has been appointed Instructor; Assistant Professor C. G. A. Nordling has been promoted to Associate Professor.

University of Oklahoma: Assistant Professors R. C. Dragoo and J. B. Giever have been promoted to Associate Professors; Professor Casper Goffman is on leave for one year as Visiting Professor at Purdue University.

University of Oregon: Associate Professor F. C. Andrews, University of Nebraska, has been appointed Associate Professor; Assistant Professor F. W. Anderson, University of Nebraska, has been appointed Assistant Professor; Mr. J. A. Dubay, Graduate Student, University of Chicago, has been appointed Instructor; Associate Professor Paul Civin has been promoted to Professor; Dr. L. W. Anderson has been promoted to Assistant Professor.

University of Pennsylvania: Dr. Morikuni Goto, Tulane University, has been appointed Assistant Professor; Dr. Robert Ellis, Pennsylvania State University, has been appointed Visiting Assistant Professor; Dr. Mandakini Rohatgi has been appointed Instructor; Dr. J. J. Price has been appointed Lecturer; Associate Professor Bernard Epstein is on leave during 1957–58 at Stanford University; Assistant Professor Murray Gerstenhaber is on leave during 1957–58 at the Institute for Advanced Study.

University of Saskatchewan: Visiting Associate Professor O. P. Aggarwal, University of Alberta, has been appointed Special Lecturer; Professor Peter Scherk who served as Visiting Professor, University of Pennsylvania, during 1956–57, has returned.

University of South Carolina: Assistant Professor R. Z. Vause, University of Kansas City, has been appointed Assistant Professor; Professor W. L. Williams has been elected a Fellow in the American Association for the Advancement of Science.

University of Tennessee: Mrs. Mary F. Dubose and Mr. R. P. Savage have been appointed Instructors, Professor A. A. Grau, University of Oklahoma, and Professor Ky Fan, University of Notre Dame, have been appointed Lecturers; Mr. P. H. Doyle has been promoted to Acting Assistant Professor; Professor O. G. Harrold, Jr., has received a Guggenheim Fellowship and is on leave of absence at the Institute for Advanced Study and Oxford University, England.

University of Texas: Dr. Alfred Schild, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, has been appointed Professor; Associate Professor R. E. Greenwood has been promoted to Professor; Dr. R. C. Osborn and Dr. Richard Kelisky have been promoted to Assistant Professors.

University of Tulsa: Mr. W. B. Garrison, Douglas Aircraft Company, Tulsa, Oklahoma, has been appointed Assistant Professor; Mr. R. G. Laatsch, Graduate Assistant, University of Missouri, has been appointed Instructor.

University of Washington: Dr. H. J. Bremermann, Member, Institute for Advanced Study, Research Assistant Professor J. M. G. Fell of the University, Dr. J. R. Isbell, Tulane University, Assistant Professor J. P. Jans, Ohio State University, and Dr. R. A. Macauley, University of California, have been appointed Assistant Professors; Dr. H. S. Baar, Teaching Associate, University of California, Dr. J. M. Gonzalez-Fernandez, Teaching Assistant, Northwestern University, and Dr. August Newlander, Teaching Assistant, Institute of Mathematical Sciences, have been appointed Instructors; Mr. K. R. Stromberg, Research Assistant at the University, has been appointed Acting Instructor; Associate Professors V. L. Klee and D. G. Chapman have been promoted to Professors; Dr. R. M. Blumenthal has been promoted to Assistant Professor; Professor R. A. Beaumont is Acting Executive Officer during 1957–58; Professor M. G. Arsove has received a Fulbright Award and is on leave in Paris, France; Professor J. H. Walter has received a National Science Foundation Postdoctoral Fellowship and is spending the year at the Institute for Advanced Study.

University of Wisconsin: Mr. B. L. Foster, University of Washington, Dr. J. M. Osborn, Teaching Assistant, University of Connecticut, and Dr. E. C. Posner, Graduate Student, University of Chicago, have been appointed Instructors; Assistant Professor C. W. Curtis has been promoted to Associate Professor; Professor R. H. Bing is on leave during 1957–58 as Visiting Professor at the Institute for Advanced Study.

University of Wyoming: Miss Esther E. Guerin, Teacher, Roxbury Township High School, New Jersey, has been appointed Instructor; Assistant Professor W. N. Smith has been promoted to Associate Professor.

Western Illinois University: Dr. G. H. Miller has been appointed Associate Professor; Mr. R. L. Suhr has been appointed Instructor.

Western Michigan University: Assistant Professor H. F. Becksfort, Ohio University, and Mr. Stanislaw Leja, Cornell University, have been appointed Assistant Professors; Assistant Professor J. H. Powell has been promoted to Associate Professor.

West Virginia University: Mrs. Gladis Loehr, Arkansas Agricultural and Mechanical College, and Captain R. C. Strong (USN Ret.) have been appointed Instructors; Assistant Professor I. D. Peters has been promoted to Associate Professor; Mr. C. N. Cochran has been promoted to Assistant Professor.

Wisconsin State College, River Falls: Mr. R. W. Willson, State University of Iowa, has been appointed Instructor; Mr. G. D. Bisbey has been promoted to Assistant Professor.

Mr. Walter Abramowitz, Analytical Engineer, Pratt & Whitney Aircraft, East Hartford, Connecticut, has a position as Mathematician at the Control Instrument Company, Brooklyn, New York.

Dr. W. E. Barnes, State College of Washington, has been promoted to Assistant Professor.

Dr. J. C. Bradford, formerly Graduate Assistant, University of Oklahoma, has been appointed Assistant Professor, Abilene Christian College.

Dean W. H. Bradford, McNeese State College, has a position as a staff member with the Sandia Corporation, Albuquerque, New Mexico.

Dr. Barron Brainerd, University of British Columbia, has been appointed Assistant Professor at the University of Western Ontario.

Mr. F. C. Calabrese, Junior Development Engineer, Goodyear Aircraft Corporation, Akron, Ohio, has been appointed Associate Engineer, Special Engineering Products Division, International Business Machines Corporation, Poughkeepsie, New York.

Assistant Professor Jean M. Calloway, Carleton College, has been promoted to Associate Professor.

Professor E. L. Canfield, Drake University, has been appointed Dean of the Graduate Division.

Dr. D. I. Caplan, Staff Consultant, Burroughs Corporation, Philadelphia, Pennsylvania, has been appointed Chief Engineer, Navigation Computer Corporation, Philadelphia.

Assistant Professor K. H. Carlson, Valparaiso University, has been promoted to Associate Professor and Chairman of the Department of Mathematics.

Mr. C. R. Carr, Indiana Technical College, has been promoted to Associate Professor.

Assistant Professor W. L. Carter, University of Cincinnati, has been promoted to Associate Professor of Education.

Professor F. L. Celauro, East Tennessee State College, has been appointed Professor at Central Michigan College.

Mr. F. A. Ceney, Jr., Teacher, East Peoria Community High School, Illinois, is teaching at Hillsboro High School, Illinois.

Mr. D. F. Clapp, Graduate Student, University of Technology, Toronto, Ontario, has a position as a staff member at the Lincoln Laboratory, Massachusetts Institute of Technology, Lexington, Massachusetts.

Professor C. L. Clark, Oregon State College, has been appointed Head of the Department of Mathematics, Los Angeles State College.

Mr. G. M. Clough, Mathematician, Battelle Memorial Institute, Columbus, Ohio, has a position as Engineer at Shell Pipe Line Corporation, Houston, Texas.

Mr. P. J. Cocuzza, Graduate Student, Brooklyn College, has a position as Junior Electrical Engineer for the City of New York, Department of Public Works.

Mr. W. J. Cody, Jr., Graduate Assistant at the University of Oklahoma, has been promoted to Instructor.

Mr. S. H. Cohn, Aerodynamicist, AVRO Aircraft Limited, Malton, Ontario, Canada, is now a senior computing specialist.

Mr. W. L. Congleton has a position as a member of the technical staff, Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. G. F. Cramer, Mathematician, Remington-Rand, New York, New York, is Consultant for IBM Research Laboratory, Poughkeepsie, New York.

Dr. T. H. M. Crampton, Mount Holyoke College, has been promoted to Assistant Professor.

Dr. Arno Cronheim, Brandeis University, has been appointed Instructor at Ohio State University.

Dr. C. H. Cunkle, Senior Research Engineer, Convair, San Diego, California, has been appointed Assistant Professor at Colorado State University.

Mr. C. F. Daniel is employed as Research Engineer by North American Aviation "Autonetics," Downey, California.

Dr. H. J. Dark, Chairman of the Department of Mathematics, David Lipscomb College, has been appointed Associate Professor at Middle Tennessee State College.

Associate Professor Marguerite D. Darkow, Hunter College, has been promoted to Professor.

Mr. H. J. Davis, Pomona College, has been appointed Instructor at Long Beach State College.

Mr. Sol. Davis, Mathematician, Naval Air Material Center, Philadelphia, Pennsylvania, has a position as Vibration Engineer, General Electric Company, Philadelphia.

Assistant Professor F. C. DeSua, University of Pittsburgh, has a position as a member of the technical staff, Bell Telephone Laboratories, Whippany, New Jersey.

Mr. R. E. Dowd, Garnac Grain Company, New York, New York, is now Propulsion Engineer, Grumman Aircraft Engineering Corporation, Bethpage, New York.

Mr. S. S. Draeger, Pan American College, has been promoted to Dean of the School of Technology.

Assistant Professor D. E. Edmondson, Southern Methodist University, has been promoted to Associate Professor.

Dr. J. L. Ericksen, Mathematician, Naval Research Laboratory, Washington, D. C., has been appointed Associate Professor of Theoretical Mechanics, Johns Hopkins University.

Mr. H. L. Farris, Mathematician, Douglas Aircraft Company, Tulsa, Oklahoma, has a position as Mathematician at Black, Sivalls, & Bryson, Tulsa.

Mr. G. F. Feeman, Muhlenberg College, is on leave of absence as Danforth Teacher, Danforth Foundation, St. Louis, Missouri.

Assistant Professor A. B. Finkelstein, Long Island University, has been appointed Associate Professor at Pratt Institute.

Associate Professor Walter Fleming, Mankato State Teachers College, has been appointed Professor and Head of the Department of Mathematics, Hamline University.

Dr. H. J. Fletcher, Brigham Young University, has been promoted to Associate Professor.

Mr. A. J. Flynn, Engineering Assistant, General Electric Company, Bloomington, Illinois, has a position as Computing Technician, Remington Rand UNIVAC, Chicago.

Dr. J. E. Forbes, Purdue University, has been promoted to Assistant Professor.

Mr. Allen Fox has accepted a position as Electrical Engineer, IBM Product Development Laboratory, Poughkeepsie, New York.

Mr. G. C. Fraser, Assistant Director, United Nations Program of American Friends Service Committee, New York, New York, has a position at the George School, Bucks County, Pennsylvania.

Assistant Professor N. S. Free, Rensselaer Polytechnic Institute, has been promoted to Associate Professor.

Dr. T. C. Fry, Retired Assistant to Director, Bell Telephone Laboratories, New York, New York, is Vice President, Sperry Rand Corporation, Remington Rand Division, Stamford, Connecticut.

Mr. B. A. Fusaro, Middlebury College, has been appointed Instructor, University of Maryland.

Mr. C. L. Gape, University of Buffalo, has been appointed Research Assistant, Syracuse University.

Mr. F. W. Gibson, Research Engineer, Radioplane Company, Van Nuys, California, is Engineer, Douglas Aircraft Company, Santa Monica, California.

Associate Professor David Gilbarg, Indiana University, has been appointed Professor at Stanford University.

Dr. Seymour Ginsburg, Research Analyst, Northrop Aircraft Corporation, Hawthorne, California, has a position as Senior Research Engineer at National Cash Register Company, Hawthorne.

Associate Professor A. M. Gleason, Harvard University, has been promoted to Professor.

Mr. W. H. Glenn, Jr., Assistant Principal, John Muir High School, Pasadena, California, has been appointed Assistant Coordinator, Mathematics, Pasadena City Schools.

Mr. Isidore Goldman, Laboratory Director, Weba, New Hyde Park, New York, is Project Engineer, American Machine and Foundry Company, Brooklyn, New York.

Mr. E. G. Goman, College of Puget Sound, has been promoted to Associate Professor and Chairman of the Department of Mathematics.

Assistant Professor W. T. Graybeal, Emory and Henry College, has been promoted to Associate Professor.

Assistant Professor L. J. Green, Case Institute of Technology, has been promoted to Associate Professor.

Mr. Arnold Grudin, Assistant, Syracuse University, has been appointed Assistant Professor, Denison University.

Mr. W. J. Harrold, Research Fellow, Polytechnic Institute of Brooklyn, has a position as Associate Engineer, Sperry Gyroscope Company, Great Neck, New York.

Professor G. E. Hay, University of Michigan, has been appointed Acting Chairman of the Department of Mathematics.

Professor Camilla Hayden, Chairman, Department of Mathematics, St. Mary-of-the-Woods College, has been appointed Assistant Professor at the University of Toledo.

Mr. L. A. Kenna, University of Arizona, has a position with the Radio Corporation of America, Tucson, Arizona.

Dr. John Killeen, New York University, has accepted a position as Mathematician, University of California Radiation Laboratory, Livermore, California.

Assistant Professor R. M. Kozelka, University of Nebraska, has been appointed Assistant Professor, Williams College.

Dr. W. W. Leutert, Manager, Mathematical and Computer Services Department, Lockheed Missile Systems Division, is Mathematician, Electronic Computer Applications Staff, Tidewater Oil Company, San Francisco, California.

Mr. J. L. Lund, Miramonte High School, Orinda, California, has a position as Supervisor, Special Program in Teacher Education, University of California, Berkeley.

Mr. D. C. McGarvey, Yale University, has accepted a position as Mathematician, RAND Corporation, Santa Monica, California.

Associate Professor W. P. Morse, Southeastern Louisiana College, has been appointed Associate Professor at the University of Florida.

Assistant Professor J. I. Northam, Kansas State College, is employed by the Upjohn Company, Kalamazoo, Michigan.

Mr. J. A. Riley, Boston College, has a position as Research Associate, Parke Mathematical Laboratories, Carlisle, Massachusetts.

Dr. G. B. Robison, University of Connecticut, has been appointed Associate Professor, State University of New York, Teachers College at New Paltz.

Mr. R. F. Shaw, Underwood Corporation, has a position as Vice President, Digitronics Corporation, Albertson, New York.

Assistant Professor H. D. Sprinkle, University of Arizona, has a position with the Radio Corporation of America, Tucson, Arizona.

Dr. S. L. Warner, Duke University, has been promoted to Assistant Professor.

Mr. Conrad White, Teacher, Kodaikanal School, South India, has been appointed Assistant Professor, Pacific University.

Mr. C. H. Wilcox, California Institute of Technology, has been promoted to Assistant Professor.

Dean W. E. Wilson, School of Engineering, Pratt Institute, has been named Acting President of the Institute.

Mr. B. H. Youell, West Virginia University, has been appointed Instructor, West Virginia Institute of Technology.

Mr. A. F. Bentley, Paoli, Indiana, died on May 21, 1957. He was a member of the Association for nine years.

Professor H. L. Black, North Dakota Agricultural College, died in February, 1957. He was a member of the Association for fifteen years.

President Emeritus E. O. Lovett, Rice Institute, died on August 13, 1957. He was a charter member of the Association.

Mr. C. C. Massimiano, Head, Department of Mathematics, Pittsfield High School, Massachusetts, died on March 1, 1957.

Lt. L. L. McCelvey, United States Air Force, died on August 6, 1957.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

TELEVISION AND FILMS

It is proposed to collect in one issue of the MONTHLY as many articles as possible on the use of television and of films in the teaching of mathematics. Articles intended for this special issue should be submitted to the Editor, Professor R. D. James, Department of Mathematics, University of British Columbia, Vancouver 8, Canada, as soon as possible and, in any event, by March 15, 1958. Since space is limited, articles should be concise and should not include unnecessarily elaborate statistical data.

ACKNOWLEDGEMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: C. B. Allendoerfer, R. P. Boas, J. L. Brenner, R. H. Bruck, W. B. Carver, Noam Chomsky, R. R. Christian, K. L. Chung, R. V. Churchill, Paul Civin, A. H. Clifford, Nathaniel Coburn, N. A. Court, H. F. Davis, J. A. Dieudonné, L. P. Edwards, F. A. Ficken, Tomlinson Fort, J. S. Frame, E. T. Frankel, Orrin Frink Jr., F. M. C. Goodspeed, H. W. Gould, R. T. Gregory, Marshall Hall, Jr., Edwin Hewitt, Alfred Horn, G. B. Huff, T. E. Hull, S. A. Jennings, P. S. Jones, J. L. Kelley, J. G. Kemeny, F. L. Kiokemeister, M. S. Klamkin, D. H. Lehmer, Eugene Leimanis, N. H. McCoy, M. D. Marcus, Karl Menger, W. E. Milne, Leo Moser, B. N. Moyls, D. C. Murdoch, S. W. Nash, W. V. Parker, Edmund Pinney, W. V. Quine, R. A. Rankin, R. A. Restrepo, D. E. Richmond, R. A. Rosenbaum, R. L. San Soucie, A. C. Schaeffer, Peter Scherk, M. F. Smiley, A. L. Whiteman, R. M. Winger, R. J. Wisner, Max Wyman, H. J. Zassenhaus.

Mathematical Notes: C. B. Allendoerfer, T. M. Apostol, B. H. Arnold, R. A. Beaumont, R. E. Bellman, D. W. Blackett, David Blackwell, R. P. Boas, Leonard Carlitz, W. B. Carver, Paul Civin, E. A. Coddington, K. W. Crain, Arthur Erdélyi, Ky Fan, William Feller, N. J. Fine, Bernard Friedman, Casper Goffman, Michael Golomb, P. R. Halmos, O. H. Hamilton, L. A. Henkin, Fritz Herzog, Edwin Hewitt, I. I. Hirschman, A. S. Householder, T. E. Hull, J. A. Jenkins, B. W. Jones, Mark Kac, V. L. Klee Jr., Gordon Latta, D. H. Lehmer, Joseph Lehner,

D. C. Lewis, A. E. Livingston, Charles Loewner, A. T. Lonseth, C. C. MacDuffee, Wilhelm Magnus, E. A. Maier, J. E. Maxfield, Margaret Maxfield, A. F. Moursund, Zeev Nehari, C. D. Olds, Sam Perlis, J. W. Peters, W. T. Reid, Herbert Robbins, J. B. Roberts, P. C. Rosenbloom, R. L. San Soucie, W. R. Scott, R. G. Selfridge, H. N. Shapiro, Wen-Hou Shia, C. E. Springer, D. J. Struik, Olga Taussky, G. B. Thomas, Jr., L. J. Warren, M. S. Webster, L. M. Weiner, Lionel Weiss, L. B. Williams.

Classroom Notes: C. B. Allendoerfer, R. W. Ball, P. T. Bateman, E. E. Betz, B. H. Bissinger, Tomlinson Fort, Samuel Goldberg, H. J. Hamilton, E. K. Haviland, R. T. Ives, R. C. James, F. B. Jones, Mark Kac, H. L. Krall, R. E. Langer, O. V. McBrien, E. M. Michalup, S. W. Nash, C. S. Ogilvy, R. C. Osborn, Francis Parker, Hans Rademacher, G. E. Raynor, D. E. Richmond, Daniel Shanks, L. J. Snell, T. H. Southard, W. L. Strother, H. S. Thurston, E. P. Vance, G. C. Vedova, R. M. Winger, R. J. Wisner, J. W. Woll.

The Editor is especially appreciative of the help given to him by the former editor, Carl Allendoerfer, and his able editorial assistant, Mrs. Helen Zuckerman, in making the transfer of editorial duties as smooth as possible. In addition, Mrs. Zuckerman undertook most of the work in connection with the publication of the two supplements to the MONTHLY in 1957.

CALENDAR OF FUTURE MEETINGS

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 30–31, 1958.

Thirty-ninth Summer Meeting, Massachusetts Institute of Technology, Cambridge, Massachusetts, August 25–28, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, May 10, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS

KENTUCKY, University of Kentucky, Lexington, April, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D.C., December 7, 1957.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1958.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Rutgers University, New Brunswick, November 1, 1958.

NORTHEASTERN

NORTHERN CALIFORNIA, San Francisco State College, January 18, 1958.

OHIO, Denison University, Granville, April, 1958.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado State College, Greeley, Spring, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April, 1958.

UPPER NEW YORK STATE, École Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

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Edited by HOWARD EVES, University of Maine
and E. P. STARKE, Rutgers University

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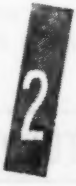

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